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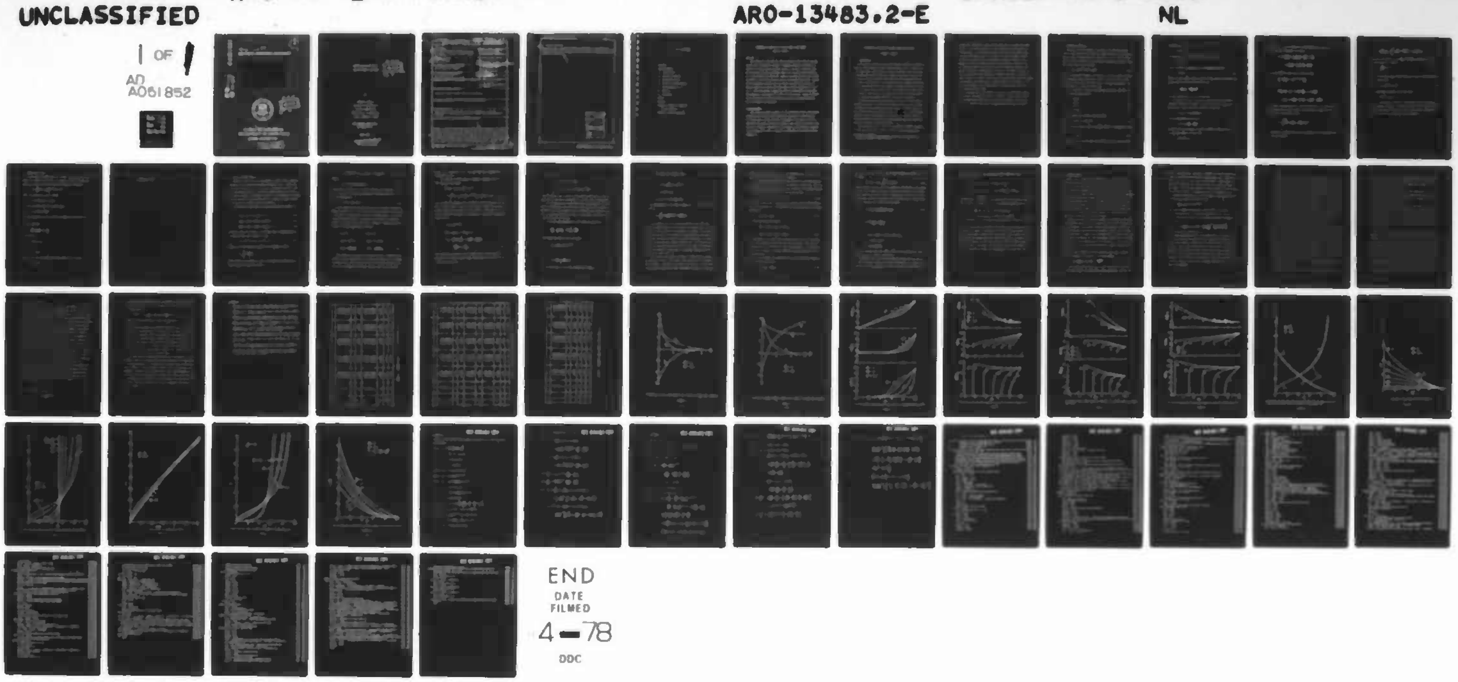

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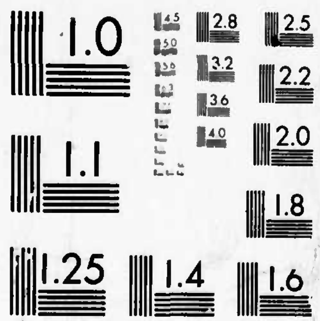

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PERTURBATION SOLUTIONS FOR VARIABLE ENERGY BLAST WAVES

by

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The trajectory of and the flow field behind blast waves with time varying energy input is determined. Freeman's Lagrangean coordinate formulation is modified to include both the geometric factor, alpha — α, for plane, cylindrical and spherical shocks and also non-integer values of β, — beta the energy input parameter, in a single computational algorithm. Numerical problems associated with vanishing density at the fictitious piston face are then examined and solved. Second order perturbation solutions about the infinite strength shock are then obtained in Sakurai's inverse shock Mach number expansion parameter for $0 \leq \beta < \alpha + 1$.

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PERTURBATION SOLUTIONS FOR VARIABLE ENERGY BLAST WAVES

Edward T. Pitkin†

1. INTRODUCTION

The analytical theory of blast waves has been the subject of intensive study by many investigators since the pioneering work of G. I. Taylor[9] about thirty years ago. A large fraction of this work concerns waves of constant, instantaneously supplied energy which propagate and decay according to the dictates of inviscid isentropic flow behind the wave and Rankine-Hugoniot conditions at the wave front. This work has been well summarized by Sakurai[1] who has also developed an efficient method of expanding the general solution about the simpler limiting solution for the flow behind a very strong shock. Furthermore, he has shown that the square of the inverse shock Mach number is a most convenient and natural parameter to use for this expansion.

A lesser amount of work has been devoted to analysis of blast waves of variable energy input. In 1957 Lees and Kubota[6] examined this case briefly in connection with the hypersonic blast wave analogy and Rogers[3] presented the limiting strong shock solution in 1958. More recently in 1968 Freeman[2] examined this case in connection with cylindrical spark channel formation from exploding wires and Dabora[7] has applied variable blast wave techniques to droplet explosions in spray detonations.

The work reported here is basically an extension of Freeman's approach to cover the full spectrum of plane, cylindrical and spherical variable

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energy blast waves within a single analytic formulation and computation algorithm. It also contains corrections to some numerical errors discovered in Freeman's paper. As a consequence Freeman's notation has been adopted completely and derivations given in his paper have only been summarized here. His major contribution was to recast Sakurai's expansion procedure in Lagrangean coordinates, replacing distance from the origin, r, with mass, m, as an independent variable, but retaining the inverse shock Mach number as the expansion parameter. The advantage is two fold: (1) the number of differential equations that must ultimately be integrated numerically is reduced by nearly a factor of two, and (2) the position of the fictitious moving piston face which supplies the variable energy is uniquely defined to be at mass = 0. In Eulerian coordinates precise determination of the piston position can be a very difficult numerical task.

In sections 2 and 3, the equations of motion and the strong shock solution are reviewed. The expansion procedure is outlined in section 4 and a method of extrapolating the numerical solution to the mass origin is discussed in section 5. Numerical results are then summarized and discussed in section 6.

2. EQUATIONS OF MOTION

Let p_o and ρ_o be the pressure and density in the undisturbed medium while r is the radius from the origin to an arbitrary point in space and M is the total mass contained between the origin and r. Let r = R at the shock front so that the volume behind the shock front, when filled with gas of original density, will contain the mass

$$M_o = 2^{\alpha}\pi^{\alpha(3-\alpha)/2}R^{\alpha+1}\rho_o/(\alpha+1) \tag{2.1}$$

where α is a geometric parameter with values 0, 1, and 2 for plane, cylindrical and spherical waves respectively. Finally let u be the speed at any point and U be the speed of the shock front, C the speed of sound and γ the specific heat ratio in the undisturbed gas.

It is convenient to introduce non-dimensional variables. For mass, radius, velocity and density take

$$m \triangleq M/M_o \tag{2.2}$$

$$a \triangleq r/R \tag{2.3}$$

$$f \triangleq u/U \tag{2.4}$$

$$h \triangleq \rho/\rho_o \tag{2.5}$$

while the square of the inverse shock Mach number

$$y \triangleq C^2/U^2 = C^2/(dR/dt)^2 \tag{2.6}$$

may be combined with p_o to define a variable related to non-dimensional pressure

$$g = py/p_o \tag{2.7}$$

An arbitrary non-zero initial radius of the shock, R_o, is then used to define a characteristic time

$$t_o \triangleq R_o/C \tag{2.8}$$

which may then be used to non-dimensionalize time through

$$\omega \triangleq t/t_o \tag{2.9}$$

while the shock front coordinate, R, is non-dimensionalized by

$$z \triangleq R/R_o \tag{2.10}$$

Finally the decay index, a ratio of the percentage changes of shock Mach number and radius, as introduced by Sakurai[1], is

$$\lambda = \frac{(dy/y)}{(dz/z)} = \frac{-2Rd^2R/dt^2}{(dR/dt)^2}, \tag{2.11}$$

the latter obtained by use of (2.6) and (2.10).

The equations of fluid flow behind the shock wave have been derived in Lagrangean coordinates by Freeman[2]. Only the results will therefore be summarized here. First the density at any point is given by

$$h = \rho/\rho_o = 1/[(\alpha+1)a^{\alpha}\partial a/\partial m] \tag{2.12}$$

and the non-dimensional velocity is

$$f = a-(\alpha+1)m\frac{\partial a}{\partial m} + \lambda y\frac{\partial a}{\partial y}. \tag{2.13}$$

Beginning with the assumption of a purely isentropic process behind the shock for each mass element it follows that

$$\partial(p/\rho^{\gamma})/\partial y = 0 \tag{2.14}$$

which upon using the non-dimensional variables above can be shown to lead to the following in Lagrangean coordinates,

$$(\alpha+1)m\left[\left(\frac{\alpha}{a}\frac{\partial a}{\partial m}+\frac{1}{\gamma g}\frac{\partial g}{\partial m}\right)\frac{\partial a}{\partial m}+\frac{\partial^2 a}{\partial m^2}\right]+\frac{\lambda}{\gamma}\frac{\partial a}{\partial m}$$

$$-\lambda y\left[\left(\frac{\alpha}{a}\frac{\partial a}{\partial y}+\frac{1}{\gamma g}\frac{\partial g}{\partial y}\right)\frac{\partial a}{\partial m}+\frac{\partial^2 a}{\partial m\partial y}\right]=0\ . \tag{2.15}$$

Similarly Newton's second law for the mass element

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial y}\frac{dy}{dt}=-(2r)^{\alpha}\pi^{\alpha(3-\alpha)/2}\frac{\partial p}{\partial m} \tag{2.16}$$

leads to

$$\frac{a^{\alpha}(\alpha+1)}{\gamma}\frac{\partial g}{\partial m}-\frac{\lambda a}{2}+(\alpha+1)\frac{2\alpha+\lambda}{2}m\frac{\partial a}{\partial m}+(\alpha+1)^2m^2\frac{\partial^2 a}{\partial m^2}$$

$$+\lambda y\left[-2(\alpha+1)m\frac{\partial^2 a}{\partial m\partial y}+\frac{2+\lambda}{2}\frac{\partial a}{\partial y}+y\frac{\partial a}{\partial y}\frac{\partial \lambda}{\partial y}+\lambda y\frac{\partial^2 a}{\partial y^2}\right]=0\ . \tag{2.17}$$

The energy contained between the origin and the shock is the sum of the kinetic energy and the internal energy. The increment of energy over that of the same volume of undisturbed gas is

$$E=\int_0^{M_0}\left(\frac{u^2}{2}+\frac{p}{(\gamma-1)\rho}-\frac{p_0}{(\gamma-1)\rho_0}\right)dM\ . \tag{2.18}$$

Converting to non-dimensional form, using Equation (2.12) and letting a prime designate $\partial/\partial m$ gives

$$\frac{E\rho_\alpha}{(\alpha+1)p_o M_o} = \frac{1}{y}\int_0^1 \left(\frac{\gamma f^2}{2(\alpha+1)} + \frac{g a^\alpha a'}{\gamma-1}\right) dm - \frac{1}{(\gamma-1)(\alpha+1)} \quad . \tag{2.19}$$

Letting $b = 1/[(\gamma-1)(\alpha+1)]$, $E_\alpha = E\Big/\left(2\pi^{(3-\alpha)/2}\right)\alpha$ and designating the integral J after Sakurai[1] this becomes

$$\frac{E_\alpha}{p_o R^{\alpha+1}} = \frac{J}{y} - b \quad . \tag{2.20}$$

If the total energy behind the shock varies as a power of time according to the following rule

$$E_\alpha = R_o{}^{\alpha+1} p_o (t/t_o)^\beta , \tag{2.21}$$

the energy equation reduces to the simple form

$$\omega^\beta / z^{\alpha+1} = J/y - b. \tag{2.22}$$

When $\beta = 0$ the energy is constant, a case examined by many investigators and well documented by Sakurai. Letting β take non-zero values allows one to model a fairly wide range of energy inputs which can be further extended with some modification of (2.21) as has been shown by Freeman[2].

3. STRONG SHOCK LIMIT

At very large shock Mach numbers $y \to 0$ and $J/y \gg b$ so the energy equation reduces to a simple form consistent with Rogers'[3] similarity solution for the equation of motion for which the shock front advances according to

$$R = \left[R_o^{\alpha+1} C^2/n^2 J_o t_o^{\beta}\right]^{1/(\alpha+3)} t^n = Kt^n \tag{3.1}$$

where n is related to β, α and λ through

$$n = (2+\beta)/(\alpha+3) = 2/(\lambda_o+2) \tag{3.2}$$

from which the decay index for strong shocks is

$$\lambda_o = 2(\alpha+1-\beta)/(2+\beta) \tag{3.3}$$

The zero subscripts on λ and J are used to designate the strong shock case here.

Note also that

$$\frac{dz}{d\omega} = \frac{t_o}{R_o}\frac{dR}{dt} = \frac{U}{C} = \frac{1}{y^{\frac{1}{2}}} \tag{3.4}$$

while from (3.1)

$$dz/d\omega = nz/\omega \tag{3.5}$$

so that

$$z = \omega/ny^{\frac{1}{2}} \tag{3.6}$$

Combining this with the strong shock energy equation ($b = 0$) yields

$$z = z_o y^{1/\lambda_o} \tag{3.7}$$

where

$$z_o = \left(n^\beta / J_o\right)^{1/(\alpha+1-\beta)} \tag{3.8}$$

4. OUTLINE OF SERIES SOLUTION

Equations (2.15) (2.17) and (2.22) comprise a system of three dependent variables; λ, a, and g and two independent variables; m and y. Sakurai has shown that λ is a function of y only and that the pressure, density etc. are slowly varying functions of y. He therefore suggests expansion in powers of y to reduce the number of independent variables. In the Lagrangean coordinates used here this has the very desirable effect of reducing the partial differential equations to ordinary differential equations.

The expansions, taken about the strong shock solution for which y = 0, are

$$\lambda(y) = \lambda_o + \lambda_1 y + \lambda_2 y^2 + \cdots \tag{4.1}$$

$$a(m,y) = a_o(m) + a_1(m)y + a_2(m)y^2 + \cdots \tag{4.2}$$

$$g(m,y) = g_o(m) + g_1(m)y + g_2(m)y^2 + \cdots \tag{4.3}$$

By similar reasoning the equation for the shock front is expanded about the strong shock "zero order" solution (3.7) as

$$z = z_o\, y^{1/\lambda_o} (1 + z_1 y + z_2 y^2 + \cdots) \quad . \tag{4.4}$$

Equation (3.4) which applies in general can be integrated to give:

$$\omega = \int y^{\frac{1}{2}} \frac{dz}{dy}\, dy \quad .$$

Substituting for z and taking z = 0 at $\omega = 0$ then gives an expansion for the time variable as a function of y and z

$$\omega = z_o y^{(\lambda_o+2)/2\lambda_o}(\omega_o + \omega_1 z_1 y + \cdots + \omega_j z_j y^j + \cdots) \qquad (4.6)$$

where

$$\omega_j = (2j\lambda_o+2)/((2j+1)\lambda_o+2) \quad . \qquad (4.7)$$

Noting from (2.13) that $f = f(\lambda, a, m)$ it is clear that substitution of (4.1 - 4.3) into (2.19) will yield integrals in ascending powers of y so that J can be expressed as

$$J = J_o + J_1 y + J_2 y^2 + \cdot \cdot \cdot \qquad (4.8)$$

These expansions are then substituted into the isentropic, momentum and energy equations (2.15), (2.17) and (2.22) and the coefficients of like powers of y equated to give perturbation equations of ascending order. In the isentropic and momentum equations this procedure yields ordinary differential equations for a_j and g_j with λ_j entering as a parameter. These must be solved subject to the shock jump boundary conditions.

$$a_o(1) = 1 \qquad a_1(1) = 0$$

$$g_o(1) = 2\gamma/(\gamma+1) \qquad g_1(1) = (1-\gamma)/(\gamma+1)$$

$$a_o'(1) = \frac{\gamma - 1}{(\alpha+1)(\gamma+1)} \qquad a_1'(1) = \frac{2}{(\alpha+1)(\gamma+1)} \qquad (4.9)$$

The remaining coefficients are all zero at the shock. Fortunately the successive orders of approximation to the isentropic equation yield intermediate integrals for g_j. These may then be used in the momentum equation which must be solved by numerical integration from the shock front to the mass origin. The equations and solutions for each order will be discussed in the following sections.

Now consider the energy equation. Putting the expansion (4.4), (4.6) and (4.8) into (2.22) gives

$$z_o^{\beta}\,(\omega_o + \omega_1 z_1 y + \omega_2 z_2 y^2 + \cdots)^{\beta}$$

$$= z_o^{\alpha+1}(1 + z_1 y + z_2 y^2 + \cdots)^{\alpha+1}(J_o + (J_1 - b)y + J_2 y^2 + \cdots). \quad (4.10)$$

The two series with exponents may be converted to simple power series by means of the binomial formula if the first term of each is larger than the sum of the remaining terms. This is certainly true for small y. Multiplying out on the right and equating coefficients of like powers of y then yields relations between J_j and z_j:

$$J_o = \omega_o^{\beta} z_o^{\beta-(\alpha+1)},$$

$$J_1/J_o = \left[\beta\omega_1/\omega_o - (\alpha+1)\right] z_1 + b,$$

$$J_2/J_o = \left[\frac{\beta(\beta-1)}{2}\frac{\omega_1^2}{\omega_o^2} + (\alpha+1)\left(\frac{\alpha+2}{2} - \frac{\beta\omega_1}{\omega_o}\right)\right] z_1^2$$

$$+ \left[\frac{\beta\omega_2}{\omega_o} - (\alpha+1)\right] z_2 \,. \quad (4.11)$$

These relations are equally valid for integer and non integer values of β. Freeman[2] only considered integer values. Substituting (4.4) into (2.11) and solving for the z_j gives

$$z_1 = -\lambda_1/\lambda_o^2$$

$$z_2 = -[\lambda_2 + (1+\lambda_o)z_1\lambda_1]/2\lambda_o^2$$

$$z_3 = -[\lambda_3 + (1+\lambda_o)z_1\lambda_2 + (1+2\lambda_o)z_2\lambda_1]/3\lambda_o^2 \qquad (4.12)$$

$$\vdots \qquad \vdots$$

which may then be used in (4.11) to give relations between the λ_j and J_j. These will be used later to evaluate the λ_j for each order of approximation.

The solution proceeds in ascending orders of approximation. The zeroth order, corresponding to the strong shock solution with $y \to 0$, is used in the computation of the first order solution and the first order solution is then used to compute the second and so forth. In this paper the solution will be carried only to the 2nd order.

The zeroth order approximation to the isentropic equation is

$$\frac{g_o'}{g_o} = -\left[\frac{\lambda_o}{(\alpha+1)m} + \gamma\left(\alpha\frac{a_o'}{a_o} + \frac{\alpha_o''}{\alpha_o'}\right)\right] \qquad (4.13)$$

which Freeman has shown to have an integral

$$g_o = K_o \Big/ (a_o{}^{\alpha} a_o')^{\gamma} m^{\lambda_o/(\alpha+1)} \qquad (4.14)$$

where

$$K_o = \frac{2\gamma}{\gamma+1}\left[\frac{\gamma-1}{(\alpha+1)(\gamma+1)}\right]^{\gamma} \qquad (4.15)$$

The momentum equation in the zeroth approximation becomes

$$a_o'' = (B_o a_o' - C_o a_o - g_o a_o{}^{\alpha} \lambda_o / m)/A_o \,, \tag{4.16}$$

where

$$A_o = (\alpha+1)\gamma\,[a_o^{\alpha} g_o / a_o' - (\alpha+1)m^2] \,,$$

$$B_o = (\alpha+1)m\gamma(\alpha+\lambda_o/2) \,,$$

$$C_o = \gamma[\lambda_o/2 + \alpha(\alpha+1)g_o a_o^{\alpha} a_o'/a_o^2] \,. \tag{4.17}$$

The first approximation to the energy integral, J, of (4.8) will be obtained by evaluating

$$J_o = \int_0^1 \left[\frac{\gamma}{2}\,\frac{(a_o-(\alpha+1)ma_o')^2}{\alpha+1} + \frac{g_o a_o^{\alpha} a_o'}{\gamma-1}\right] dm \tag{4.18}$$

along with integration of (4.16) from the shock front to the mass origin. (Note that Freeman's forms of this and higher order integrals are only correct for $\alpha = 1$.) The first term of this expression is a measure of the kinetic energy behind the shock and the second term is proportional to the internal energy so these two factors can easily be accumulated separately. The integration of a_o is initiated with the boundary conditions (4.9). The values of $a_o(m)$, $a_o'(m)$, $g_o(m)$ and $g_o'(m)$ may be stored for use in the first order solution or, more conveniently, these integrations can be performed simultaneously with those for the higher order solutions.

In the zeroth order equation λ_o appears as a known parameter obtained from the similarity solution. In the higher order equations the λ_j's appear as unknown parameters. Fortunately, the equations are linear in λ_j and of course the perturbation equations are linear differential equations so it follows that superposition of solutions is valid. This property will allow evaluation of the λ_j's after the equations have been integrated.

In each higher order approximation the g_j' term in the momentum equation may be eliminated with the isentropic equation and its integral, g_j. The integrals have been obtained through third order by Freeman. The momentum equation then gives higher order perturbation equations of the form

$$a_j'' = (B_j\, a_j' + C_j\, a_j + D_j + \lambda_j\, E_j)/A_o \tag{4.19}$$

in which the coefficients B_j through E_j are functions only of known quantities from lower order solutions. It follows that a_j can be expressed as the linear combination

$$a_j = a_{j1} + \lambda_j a_{j2} \tag{4.20}$$

so that (4.19) gives two equations

$$a_{j1}'' = (B_j\, a_{j1}' + C_j\, a_{j1} + D_j)/A_o \;,$$

$$a_{j2}'' = (B_j\, a_{j2}' + C_j\, a_{j2} + E_j)/A_o \;. \tag{4.21}$$

It is convenient to let the non-zero boundary condition of (4.2) be satisfied by a_{11} so the remaining perturbations are zero at the shock. Note that the denominator, A_o, is the same for all orders of approximation. This term goes to zero at $m = 0$ when $\beta = 0$ causing computational difficulties which will be discussed later.

It will be evident upon substitution of a_j into the isentropic equation that g_j must be of the form

$$g_j = g_{j1} + \lambda_j\, g_{j2} \;. \tag{4.22}$$

Finally substituting a_j and g_j into the energy integral will show J_j to be of the form

$$J_j = J_{j1} + \lambda_j J_{j2} = \int_0^1 (J'_{j1} + \lambda_j J'_{j2})dm \tag{4.23}$$

where the terms J_{j1} and J_{j2} are to be accumulated simultaneously with integration of (4.21). When J_{j1} and J_{j2} have thus been evaluated λ_j can be determined. After using (4.12) in (4.11) to eliminate z_j in favor of λ_j, (4.23) is substituted in to the left hand side of (4.11) and the result is then solved for λ_j. The first two of these coefficients are

$$\lambda_1 = \frac{b - J_{11}}{J_{12} + J_o[\omega_1\beta/\omega_o - (\alpha+1)]/\lambda_o^2} \tag{4.24}$$

and

$$\lambda_2 = \frac{J_o\ \lambda_1^2\ G - 2\lambda_o^2\ J_{21}}{J_o\ H - 2\lambda_o^2\ J_{22}} \tag{4.25}$$

where

$$G = \beta(\beta-1)\ \omega_1^2/\omega_o^2 + (\alpha+1)(\alpha+1-\lambda_o-2\beta\omega_1/\omega_o)$$

$$+ (1+\lambda_o)\ \beta\ \omega_2/\omega_o$$

$$H = \beta(\omega_2/\omega_o) - (\alpha+1) \tag{4.26}$$

The ω_j are obtained from (4.7). Expressions for the coefficients A_j through E_j, J'_{j1} and J'_{j2} are listed in the Appendix.

Freeman has shown that given approximations through the k'th order with $k \geq 2$, it is possible to obtain an excellent approximation to the shock position R(t) as follows. Truncate the λ series at the k+2 term and evaluate the last two λ_j in terms of those preceeding and known boundaries of the λ vs y curve at y = 1. Thus

$$\lambda(1) = \lambda_o + \lambda_1 + \cdots + \lambda_k + \lambda_{k+1} + \lambda_{k+2} = 0 \tag{4.27}$$

$$d\lambda(1)/dy = \lambda_1 + 2\lambda_2 + \cdots + k\lambda_k + (k+1)\lambda_{k+1} + (k+2)\lambda_{k+2}$$

$$= (2+\alpha)/4 \tag{4.28}$$

The higher order λ terms in (4.12) are then zero and all the z_j can be expressed in terms of λ_o through λ_{k+2}. If k = 2 the general expression for z_j when $j \geq 5$ is

$$z_j = -\Big([1+(j-4)\lambda_o]\lambda_4 z_{j-4} + [1+(j-3)\lambda_o]\lambda_3 z_{j-3}$$

$$+ [1+(j-2)\lambda_o]\lambda_2 z_{j-2} + [1+(j-1)\lambda_o]\lambda_1 z_{j-1}\Big)/j\lambda_o^2 \tag{4.29}$$

Using this and (4.12) in (4.4) and (4.6) then gives $z(\omega)$ or $R(t)$ for all values of y.

It was found that this procedure works very well for $\beta = 0$, but when β approaches $\alpha + 1$ where $\lambda \equiv 0$, satisfaction of the final slope condition (4.28) with a limited number of terms causes a false hump in the λ-y curve near y = 1. The procedure was therefore modified through multiplying λ_{k+2} as obtained by simultaneous solution of (4.27) and (4.28) by the factor $[1-\beta/(\alpha+1)]$ and solving (4.27) for λ_{k+1} once again. Thus the need to meet the final slope conditions is progressively eliminated as β approaches $\alpha + 1$ to give more realistic λ-y curves.

5. EXTRAPOLATION TO m = 0.

When $\beta = 0$ as it does in the constant energy blast wave, integration of (4.16) will show that $a_o' \to \infty$ and $a_o \to 0$ as $m \to 0$ so it follows from (4.17) that $A_o \to 0$. This latter term appears as the denominator of (4.21) so it may be anticipated that computational difficulties will be encountered near the mass origin for all orders of approximation. The problem is further compounded in evaluation of the energy integrals which contain terms proportional to a_o' and accumulate rapidly near $m = 0$. For example if $\gamma = 1.1$ and $\alpha = 1$, sixty percent of the true value of J_o is accumulated in the last one percent of mass variation. Numerical integration all the way to the origin is clearly impossible yet it is mandatory that accurate values of J_j be obtained else the next higher order approximation be seriously in error.

Freeman[2] mentioned this problem and suggested an "appropriate approximation formula" but gave no details. Similar numerical problems arising in the Eulerian formation of the problem have been noted by Sakurai[1] and Bach and Lee[4]. The procedure outlined here was developed by the author and is based upon the fact that in a region close to the mass origin the pressure term, g_o, can be taken to be constant to a very high degree of approximation. Although the procedure is only necessary when β is close to zero, it is still valid for larger values of β so it can be incorporated into a computer program valid for all values of β.

Consider the integrand of (4.18) in which the first term is related to the kinetic energy and the second to the internal energy.

$$J_o' = \frac{\gamma}{2(\alpha+1)}\left(a_o-(\alpha+1)ma_o'\right)^2 - \frac{g_o a_o^\alpha a_o'}{\gamma-1} \qquad (5.1)$$

In the region near the origin, say $m < 10^{-3}$, the density is very low so it may be anticipated - and verified by numerical experiment - that the first term is much smaller than the second. Furthermore it has been observed from numerical integrations that the product ma_o' approaches zero at nearly the same rate as a_o so that this first term is nearly constant. These numerical integrations also showed g_o to be constant to about four significant figures in this region. It follows that taking these factors constant over an interval of $\Delta m \leq 10^{-3}$ will introduce no significant error in the computation of J_o.

The major contributing factor to the integral is then $a_o^\alpha a_o'$ which by use of (4.14) can be expressed in terms of m as

$$a_o^\alpha a_o' = (K_o/g_o)^{1/\gamma} m^{-\lambda_o/\gamma(\alpha+1)} \tag{5.2}$$

which can be integrated analytically if g_o is taken to be constant.

The procedure then is to integrate numerically from $m = 1$ to $m = \delta$ where $0 < \delta \leq 10^{-3}$ and add a correction term obtained by analytical integration from 0 to δ, i.e.,

$$\Delta J_o = \frac{\gamma\delta}{2(\alpha+1)}(a_o-(\alpha+1)\delta a_o')^2 + \frac{g_o}{\gamma-1}\left(\frac{K_o}{g_o}\right)^{1/\gamma}\left[\frac{\delta^{1-\lambda_o/\gamma(\alpha+1)}}{1-\lambda_o/\gamma(\alpha+1)}\right] \tag{5.3}$$

where a_o and a_o' are evaluated at $m = \delta$.

Similar considerations apply to the higher order approximations to J. In each case the kinetic energy term in the integrand can be taken as constant over the last integration interval while the internal energy term can be expressed in terms of $a_o^\alpha a_o'$, g_o and essentially constant ratios such as a_j/a_o, a_j'/a_o', etc. This method has been checked by letting δ take on values from 10^{-3} to 10^{-6} and noting that all give the same final values for each J_j. A value of $\delta = 10^{-5}$ was used in the computations leading to the results reported in the next section. Formulas for the correction term ΔJ_{11}, ΔJ_{12} etc. are given in the appendix.

6. RESULTS

The equations outlined in Sections 3, 4 and the Appendix have been numerically integrated with a fourth order Runge-Kutta algorithm in double precision and extrapolated to $m = 0$ as discussed in Section 3. The procedure has been carried out to the second order in the expansion variable y (some authors[1,2,4] prefer "third approximation") for plane, cylindrical and spherical blast waves, i.e., for $\alpha = 0, 1, 2$. The range $0 \leq \beta < \alpha+1$ (above which the zeroth order similarity solution ceases to exist[6]) was covered for $\gamma = 1.1$ and 1.4. The solution for the case, $\beta = \alpha + 1$, which corresponds to the well known self similar solution for which $\lambda = 0$ and $R = Kt$, is computationally unobtainable because of a zero divisor problems evident in equation 3.7. However, solutions for β very close to this value (within .01) were obtained easily. Solutions have also been obtained for other typical values of γ for integer values of β.

Tables 1 to 3 are summaries of the expansion coefficients λ_1, λ_2 and J_o, J_1, J_2 obtained in the computation. Korobienikov and Chushkin[5], Sakurai[1], Bach and Lee[4] have all published values of λ_1 computed in Eulerian coordinates for $\beta = 0$, $\gamma = 1.4$ which agree with those given here. (Note that Bach and Lee report values of $\lambda_i/2$ while the others give values of $\lambda_i/(\alpha+1)$). The values of λ_2 and λ_1 for $\beta = 0$ and $\gamma = 1.4$ are also in agreement to the full six significant figures quoted by Bach and Lee. The values of λ_1 for $\beta = 0$ and $\gamma = 1.1$ to 3.0 are in agreement with those reported by Freeman, however his values of λ_1 for $\beta = 1$ are in error because he apparently used an incorrect value of λ_o to obtain them. His values of J_{11} and J_{12} are, however, correct. As a final check on numerical accuracy it is noted that the values of J_o reported to 4 significant figures by Rogers for $\gamma = 1.2$ and 1.4 and various β agree with values in these tables.

It can be seen from the tables that the λ coefficients approach zero rapidly as β is increased from zero. This is also shown in Figure 1. The maximum value of β is $\alpha+1$ where $\lambda = 0$ and the shock propagates with constant velocity. This drop off of the decay index implies much stronger convergence of the perturbation series as β increases.

Figure 2 shows typical variations of the J coefficients with β for $\alpha = 2$, and $\gamma = 1.4$. It is seen that J_2 drops off rapidly and J_o becomes fairly constant while J_1 takes on an increasing portion of the energy as β increases. Similar results obtain for other γ and α.

Radial pressure, density and velocity destributions are presented in Figures 3 to 5 for $\alpha = 2$, $\gamma = 1.4$ and $\beta = 0, 1, 2$. The hump in the density distubution for $y = .5$ and $\beta = 0$ is most likely due to truncation of the expansion at the third term. This suggests an upper limit of validity for $y = .5$, or a shock Mach number of 1.4, for the second order expression when $\beta = 0$. This limit can be raised considerably as β increases as in evidenced by the curves for $\beta = 1, 2$.

As β is increased from zero, the constant energy case, the flow region rapidly contracts to a thin shell between the shock and the fictitious piston located at the zero mass position as is evident in Figures 4 and 5. Even when $\beta = 0$, over 99% of the mass is located in the outer half radius of the sphere but as noted before, over 50% of the total energy can be confined to the inner 1% of the mass because the temperature becomes extremely large in this region. As β approaches $\alpha+1$ the solution approaches the self similar result for $\lambda = 0$ and y = constant. At this limit the spherical blast wave flow field occupies a thin shell within which the velocity, pressure and density are constant. Similar results obtain for other γ and α. In particular the profiles for $\alpha = 1$, $\beta = 1$ and $\gamma = 1.4$ are given in Figure 6 as replacements for similar plots given by Freeman which, as noted before, are in error due to his use of the incorrect value of λ.

The variation of the decay index, λ, and the shock position, z, with shock strength, y, is given in Figure 7 for a spherical shock with $\gamma = 1.4$ and $\beta = 0$. The decay index was obtained by use of accurate values of λ_o, λ_1 and λ_2 given in Table 2 and approximate values of λ_3 and λ_4 obtained by matching boundary conditions at $y = 1$ as outlined in section 4. The shock position was obtained from (4.6) with z_j from (4.12) and enough terms of (4.28) to insure that no truncation error would occur in the sixth significant figure. These results are compared with a decay index and shock position obtained from tables of shock position and over-pressure reported by Goldstine and Von Neumann[8] who solved the exact partial differential equations by a numerical technique. The comparison is most favorable over the whole range of y even though the internal structure of the flow field can only be predicted to $y \approx .5$ for $\beta = 0$. Of course when β becomes larger the internal approximations improve for larger y.

The effect of β on decay index and shock position is given in Figures 8 and 9 where λ and z are plotted against shock strength with β as a parameter. Note the smooth approach to $\lambda = 0$ as β approaches $\alpha+1$ in Figure 8.

It is evident in Figure 9 that a constant value of shock strength is approached in the limit $\beta = \alpha+1$ or $\beta = 3$ in this case. The value can easily be obtained by noting that a solution of the form $z \sim \omega^n$ is required in which $n = (2+\beta)/(\alpha+3) = 1$ to be consistent with the rest of the family of curves. Differentiating this relation and using (3.4), the definition of y, gives

$$dz/d\omega = z/\omega = y^{-1/2} \tag{6.1}$$

while the energy equation (2.22) gives

$$(\omega/z)^{\alpha+1} = Jy - 1/[(\alpha+1)(\gamma-1)] \tag{6.2}$$

Eliminating ω/z and using the second order approximation for J consistent with the other curves then yields the following equation which can be solved for the limiting value of y.

$$y^{(\alpha+3)/2} = J_o + \left(J_1 - 1/[(\alpha+1)(\gamma-1)]\right) y + J_2 y^2 \qquad (6.3)$$

Taking $\gamma = 1.4$, the limits are $y = 0.531$, 0.528 and 0.542 for $\alpha = 2$, 1 and 0, respectively.

The shock trajectories corresponding to Figures 8 and 9 are given in Figure 10. It is clear that there is relatively little difference in trajectories as β is varied. The major effect of increasing β is the faster decay of shock strength at low z and slower decay at large z previously shown in Figure 9. In addition there appears to be only a very weak dependence of the trajectory upon γ as the curves for $\gamma = 1.1$ fall in the same narrow band indicated for $\gamma = 1.4$. The γ effect is more evident in the z-y plots of Figure 11 which also indicate the effect of α for $\beta = 0.5$ in all cases.

The apportionment of energy between kinetic and internal modes is also of interest. These two contributions can easily be separated in the computation of the energy integral, J. The fraction of the total energy associated with kinetic energy is plotted in Figure 12 versus the shock strength with β as a parameter. These results were computed with a three term approximation of J. Note that the kinetic energy fraction increases with β while in all cases most of the energy soon is transformed to internal energy as the shock decays and y increases. The values obtained for $y = 0$ are in agreement with computations made by Rogers[3].

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β	γ	J_o	λ_1	J_1	λ_2	J_2
0	1.1	2.80158	-4.17316	-0.84575	5.91593	4.14349
	1.2	1.53186	-4.08477	-0.62865	5.09973	1.95302
	1.3	1.09904	-4.01846	-0.54156	4.57118	1.25598
	1.4	0.87708	-3.96712	-0.48974	4.20319	0.92163
	5/3	0.59873	-3.87486	-0.40999	3.61729	0.54144
	2	0.44810	-3.80867	-0.35333	3.24624	0.36366
	3	0.27579	-3.72635	-0.26385	2.82464	0.19475
.25	1.4	0.70204	-2.65379	-0.02745	2.18103	0.57057
.5	1.1	0.78106	-1.79666	3.66353	1.33644	0.59187
	1.4	0.62794	-1.72133	0.22057	0.96355	0.36104
1.0	1.1	0.60523	-0.65358	4.20887	0.07644	0.15244
	1.2	0.58859	-0.65324	1.73102	0.06742	0.14316
	1.3	0.54713	-0.65407	0.91562	0.06269	0.13736
	1.4	0.56127	-0.65559	0.51408	0.06044	0.13357
	5/3	0.53222	-0.66114	0.04626	0.06044	0.12830
	2	0.50269	-0.66886	-0.17246	0.06530	0.12565
	3	0.43705	-0.68943	-0.35263	0.08495	0.12245
1.5	1.1	0.53960	-0.18017	4.45556	-0.05095	0.02116
	1.4	0.53039	-0.18812	0.69124	-0.04452	0.03804
1.95	1.4	0.51401	-0.01130	0.80246	-0.00473	0.00116
1.99	1.4	0.51286	-0.00216	0.81088	-0.00093	-0.00091

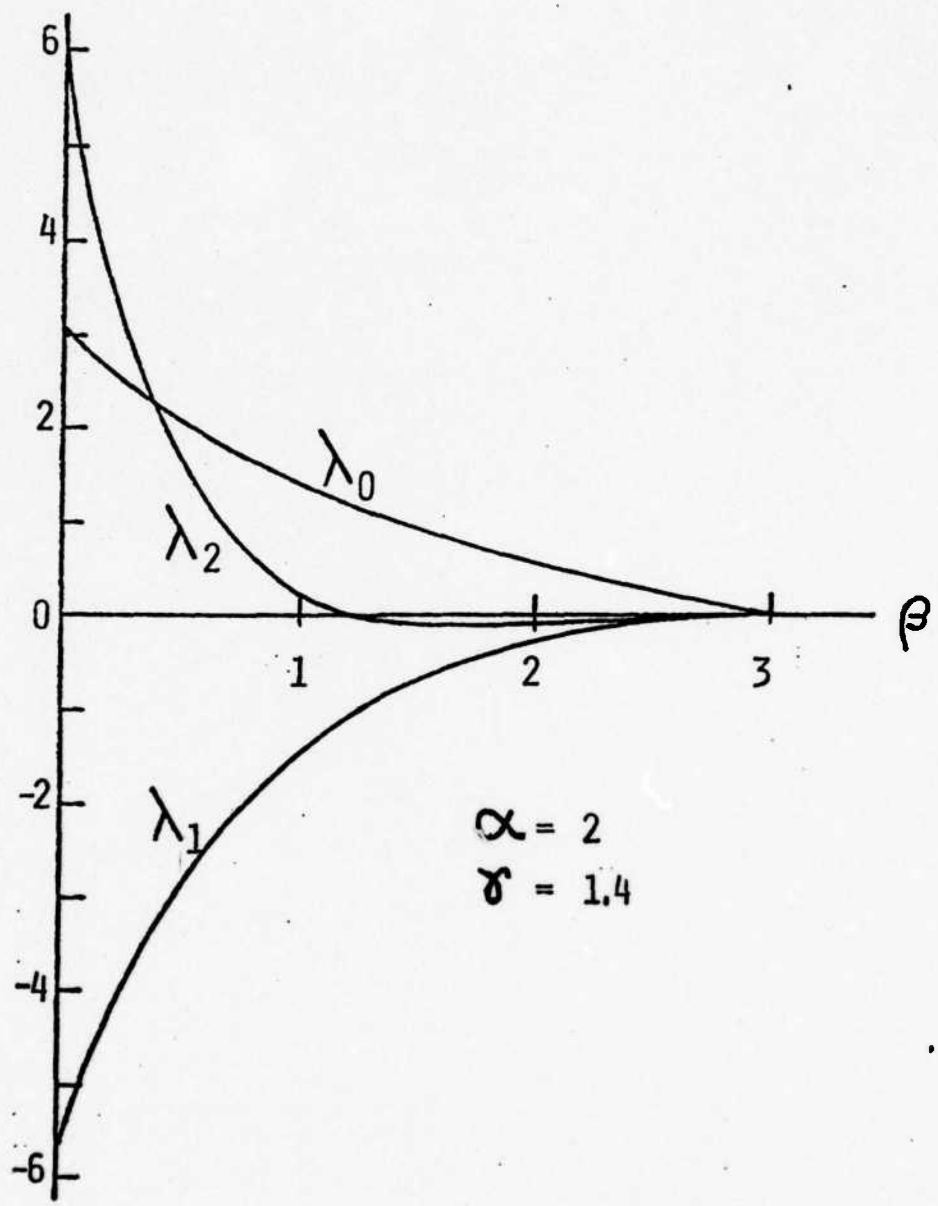
TABLE 1. EXPANSION COEFFICIENTS FOR CYLINDRICAL WAVE, $\alpha = 1$

β	γ	J_o	λ_1	J_1	λ_2	J_2
0	1.1	1.87082	-6.00300	-0.41019	7.75197	2.41710
↓	1.2	1.02644	-5.89969	-0.35189	6.80141	1.16354
↓	1.3	0.73942	-5.81883	-0.32307	6.15881	0.75899
↓	1.4	0.59261	-5.75446	-0.30338	5.69937	0.56291
↓	5/3	0.40915	-5.63556	-0.26860	4.94952	0.33752
↓	2	0.31034	-5.54866	-0.24066	4.46598	0.23100
↓	3	0.19774	-5.44223	-0.19204	3.91799	0.12912
.25	1.4	0.49105	-4.07792	-0.03462	3.14452	0.36095
.5	1.1	0.57496	-3.03230	2.35264	2.33198	0.43541
↓	1.4	0.44325	-2.87521	0.11646	1.57621	0.23932
1.0	1.1	0.44272	-1.43972	2.72249	0.34785	0.14331
↓	1.2	0.42440	-1.4241[illegible]	1.08744	0.28346	0.12362
↓	1.3	0.40974	-1.41519	0.55541	0.24441	0.11135
↓	1.4	0.39761	-1.41027	0.29596	0.21973	0.10322
↓	5/3	0.37295	-1.40748	-0.00305	0.18889	0.09144
1.5	1.1	0.39238	-0.66360	2.88715	-0.07706	0.04510
↓	1.4	0.37556	-0.66763	0.40369	-0.08871	0.04054
2.0	1.1	0.36584	-0.27807	2.98455	-0.09847	0.00554
↓	1.2	0.36489	-0.28116	1.31493	-0.09717	0.00764
↓	1.3	0.36379	-0.28403	0.75685	-0.09595	0.00959
↓	1.4	0.36257	-0.28669	0.47695	-0.09485	0.01137
↓	5/3	0.35879	-0.29290	0.13969	-0.09243	0.01542
2.5	1.1	0.34945	-0.08812	3.05043	-0.04453	-0.01159
↓	1.4	0.35401	-0.09310	0.53054	-0.04237	-0.00205
2.95	1.4	0.34846	-0.00629	0.56801	-0.00320	-0.00752
2.99	1.4	0.34804	-0.00121	0.57097	-0.00062	-0.00782

TABLE 2. EXPANSION COEFFICIENTS FOR SPHERICAL WAVE, $\alpha = 2$

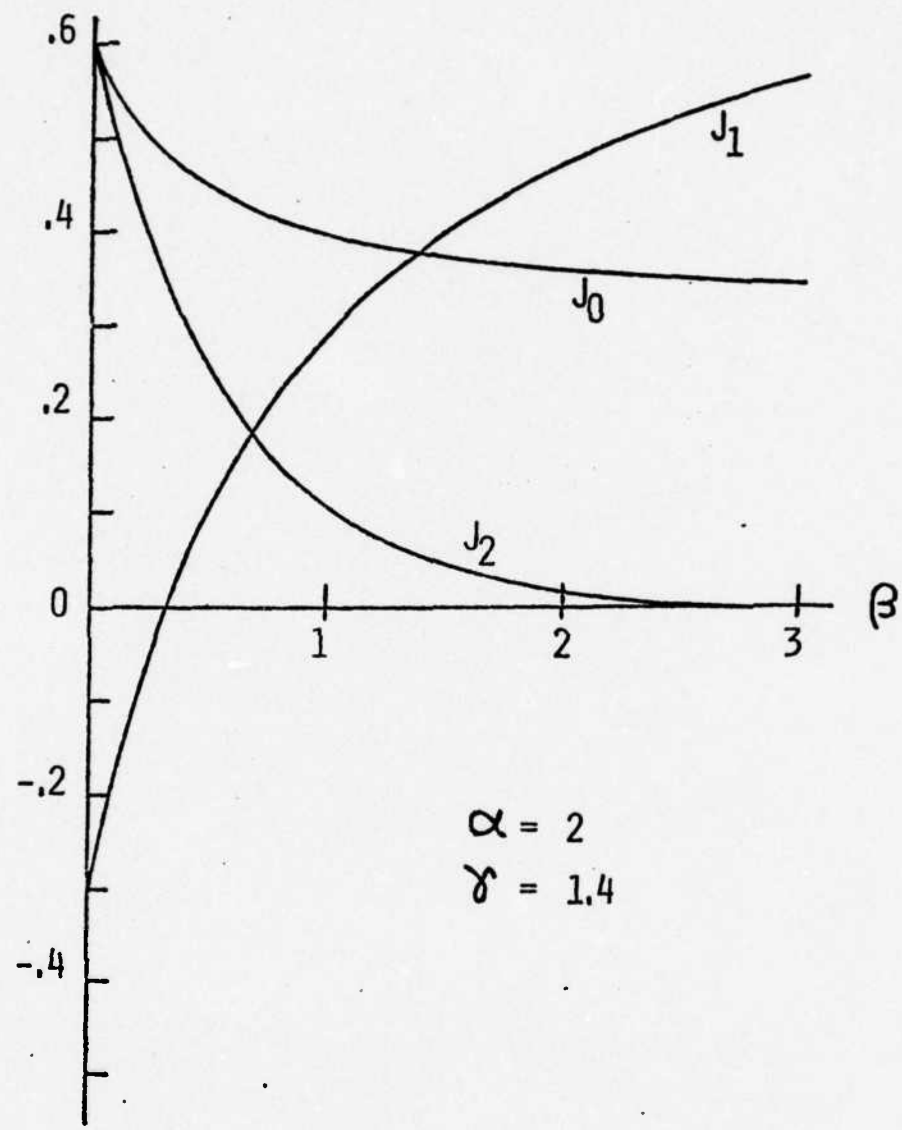
β	γ	J_o	λ_1	J_1	λ_2	J_2
0	1.1	5.57737	-2.32573	-2.97147	4.19515	11.69897
	1.2	3.02287	-2.24368	-1.78234	3.45473	5.22159
	1.3	2.14757	-2.18618	-1.36163	3.01386	3.23624
	1.4	1.69705	-2.14334	-1.13735	2.72112	2.30893
	5/3	1.13051	-2.06835	-0.83828	2.27262	1.28460
	2	0.82425	-2.01427	-0.66025	1.99397	0.82176
	3	0.47899	-1.94071	-0.42959	1.66803	0.39949
.25	1.4	1.23367	-1.20673	0.08085	1.19867	1.32576
.5	1.1	1.25640	-0.57074	7.87118	0.37492	0.91993
	1.4	1.08643	-0.56662	0.67245	0.34222	0.74140
.75	1.4	1.01463	-0.19267	1.07922	0.04087	0.36683
.95	1.4	0.97925	-0.02768	1.33264	-0.00185	0.17582
.99	1.4	0.97358	-0.00517	1.37783	-0.00061	0.14599

TABLE 3. EXPANSION COEFFICIENTS FOR PLANE WAVE, $\alpha = 0$



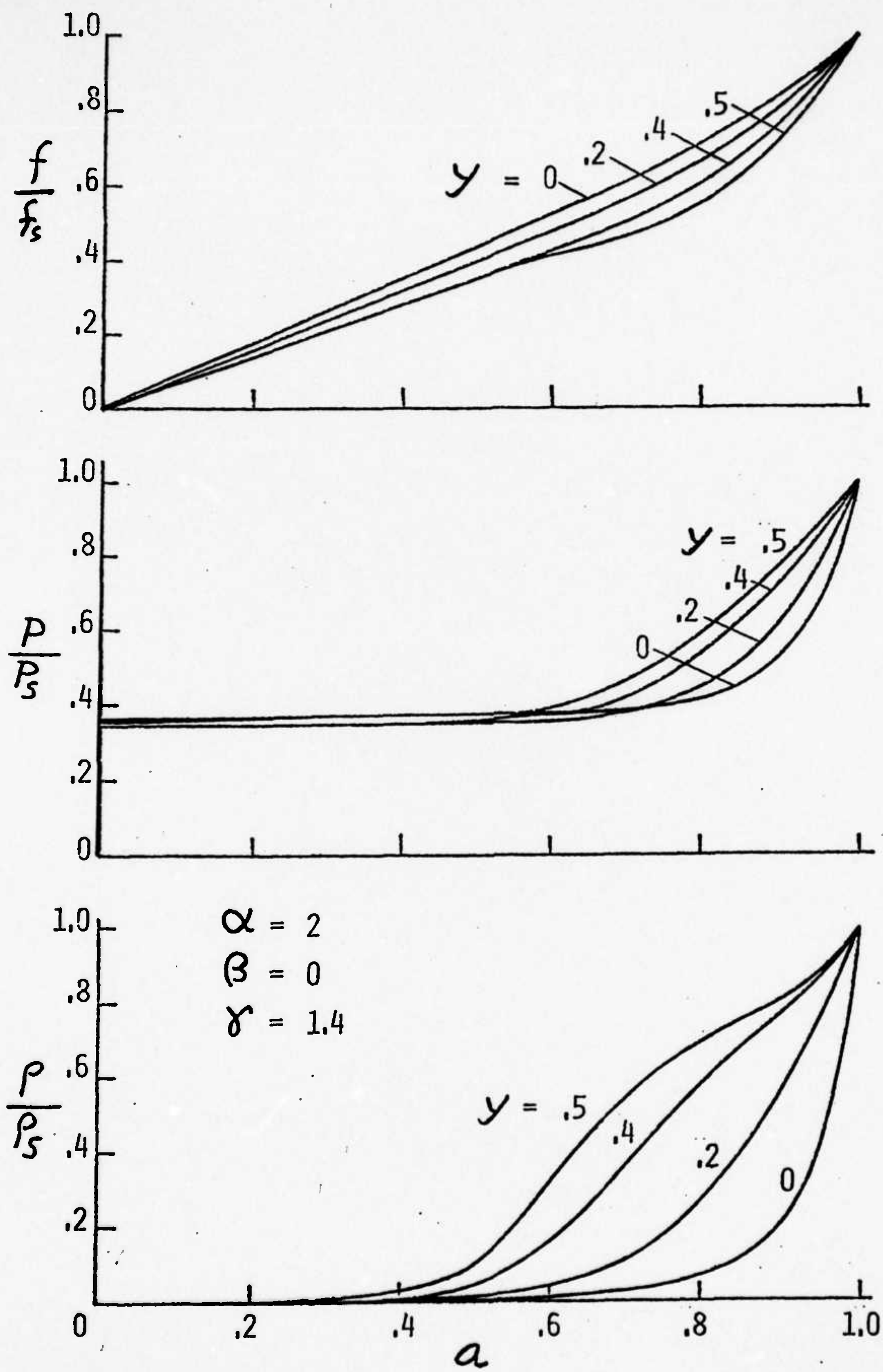
Coefficients for Decay Index Expansion as a Function of Energy Input Parameter.

Figure 1



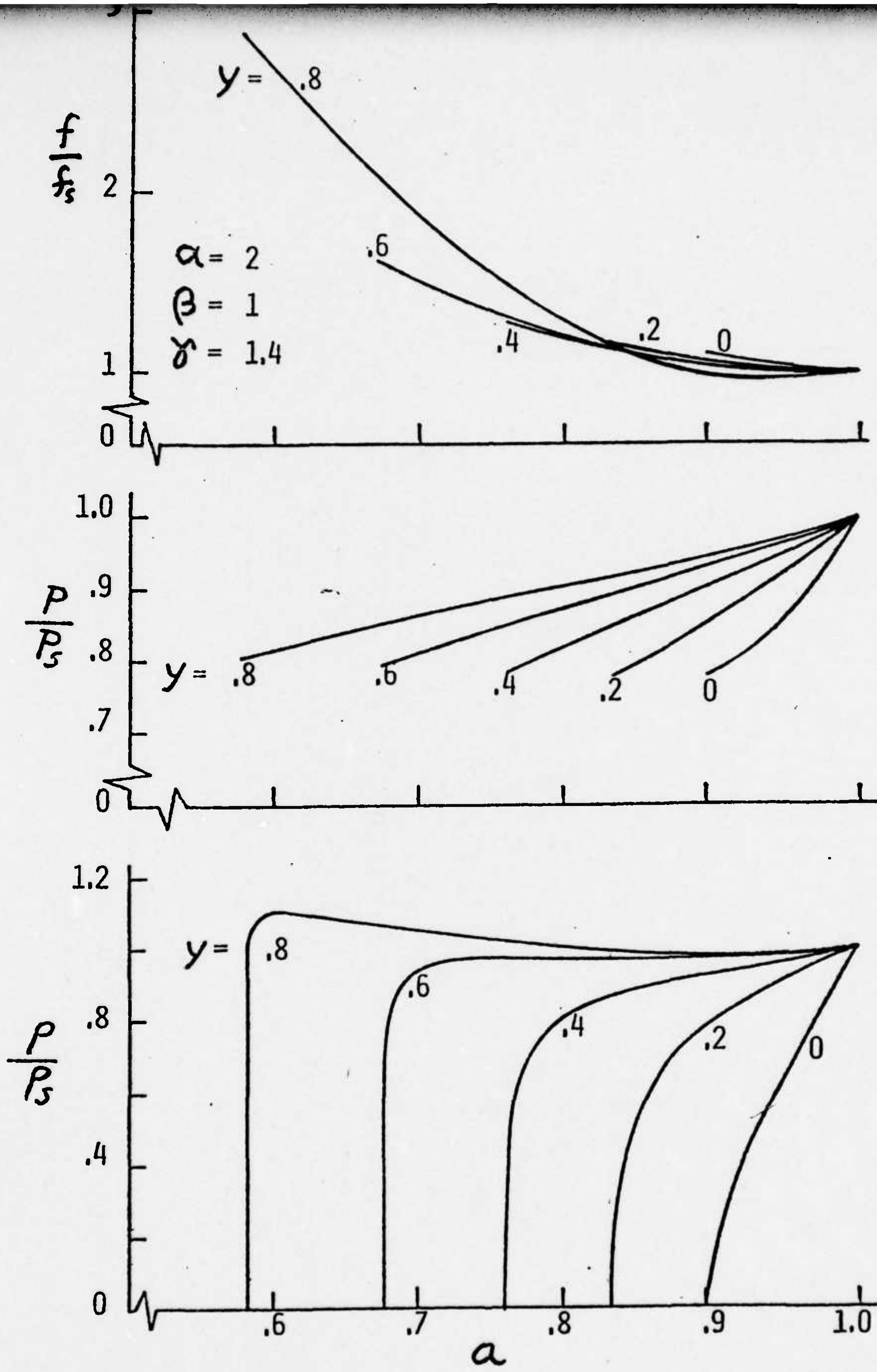
Coefficients of Energy Integral Expansion as a Function of Energy Input Parameter.

Figure 2



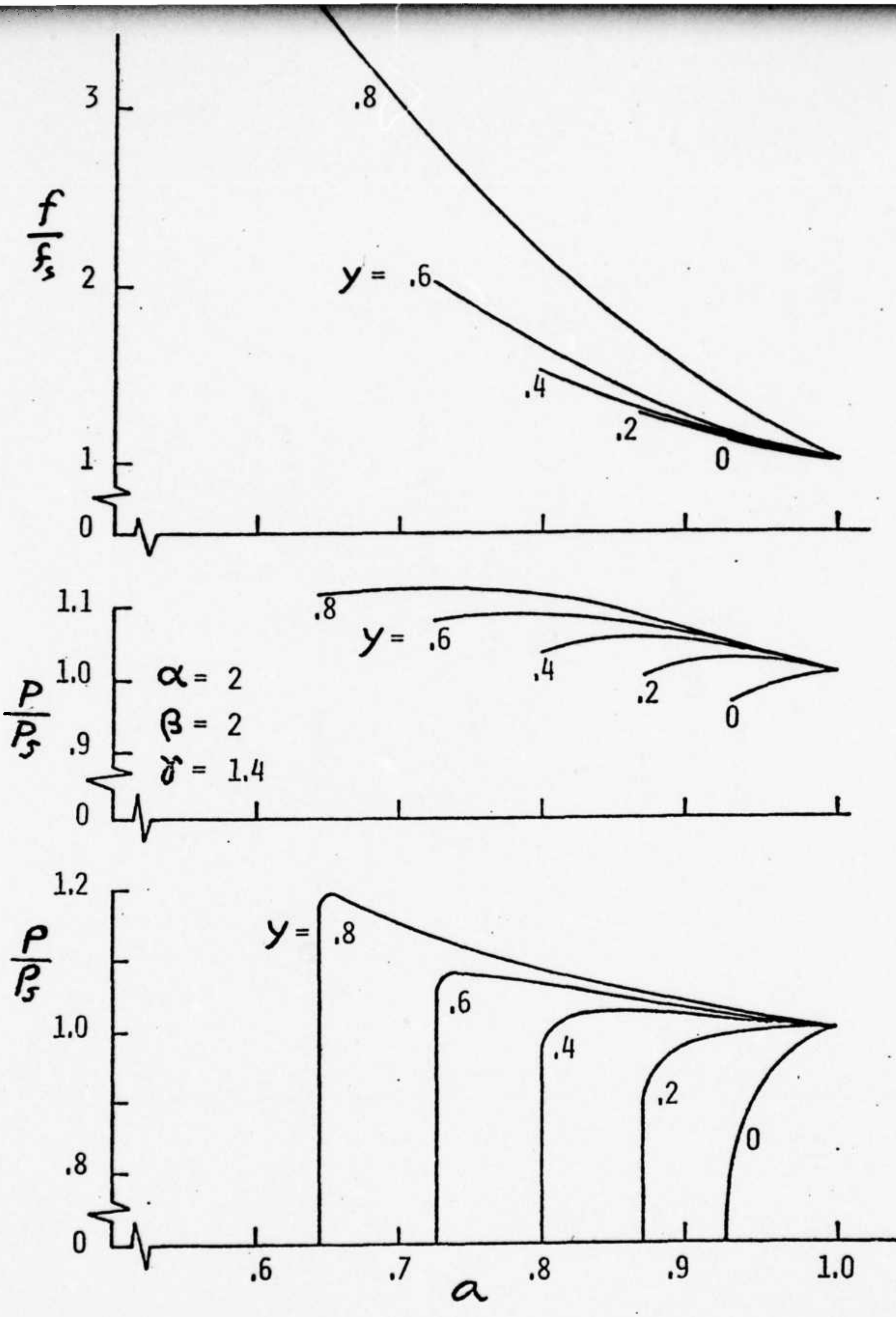
Flow Field Structure Behind Spherical Blast Wave for Instantaneous Energy Input.

Figure 3



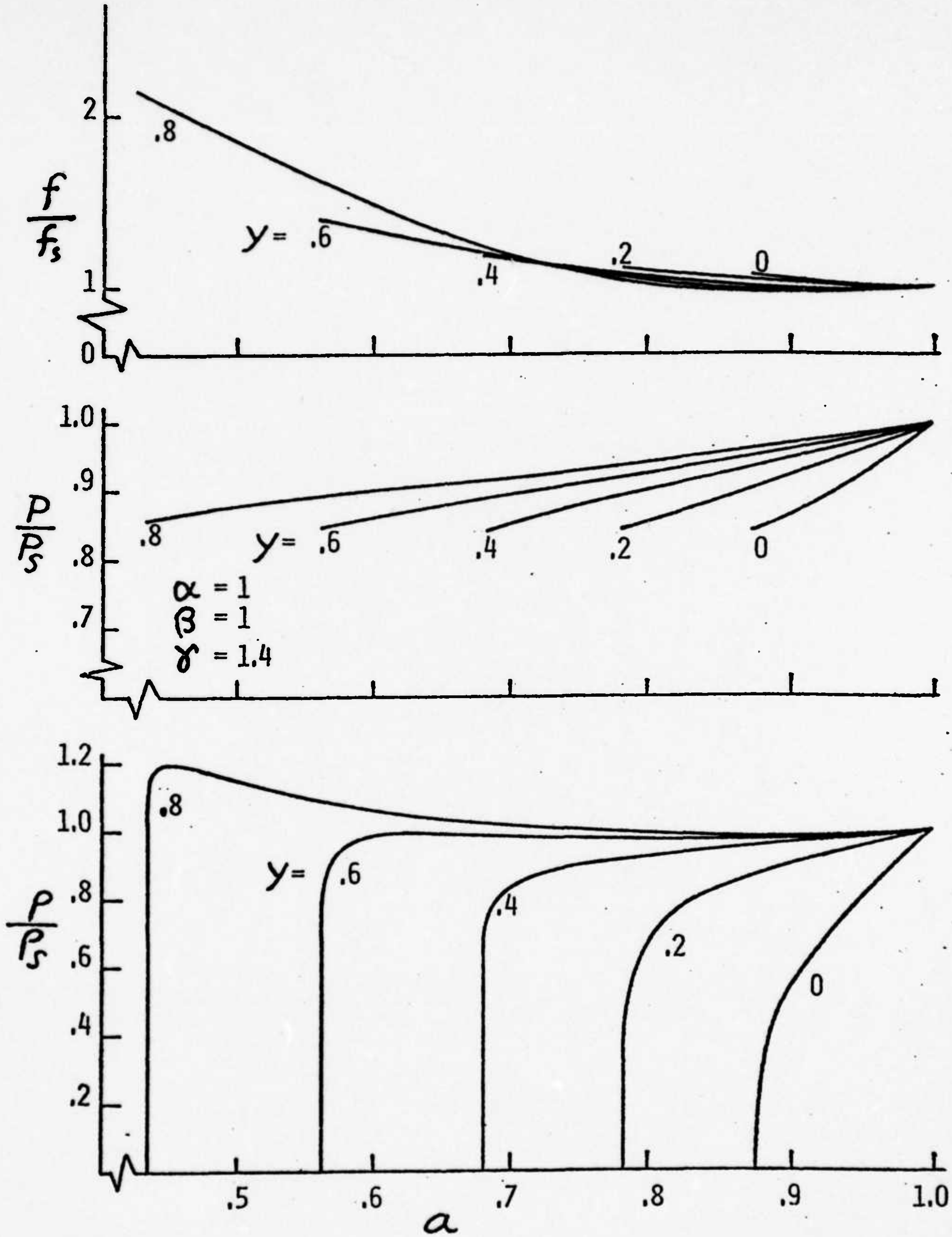
Flow Field Structure Behind Spherical Blast Wave for Constant Energy Input Rate.

Figure 4



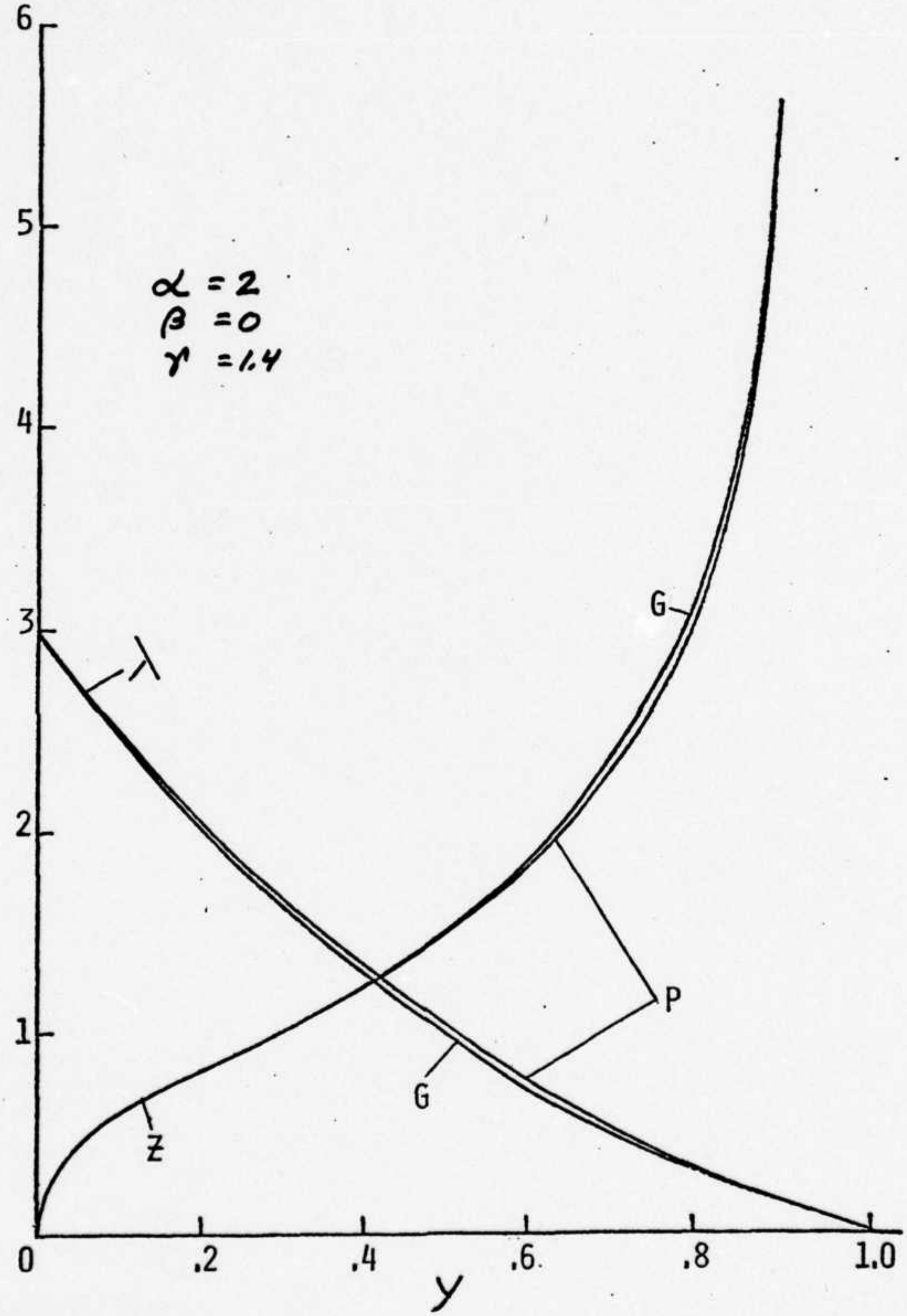
Flow Field Structure Behind Spherical Blast Wave for Linearly Increasing Energy Input Rate.

Figure 5



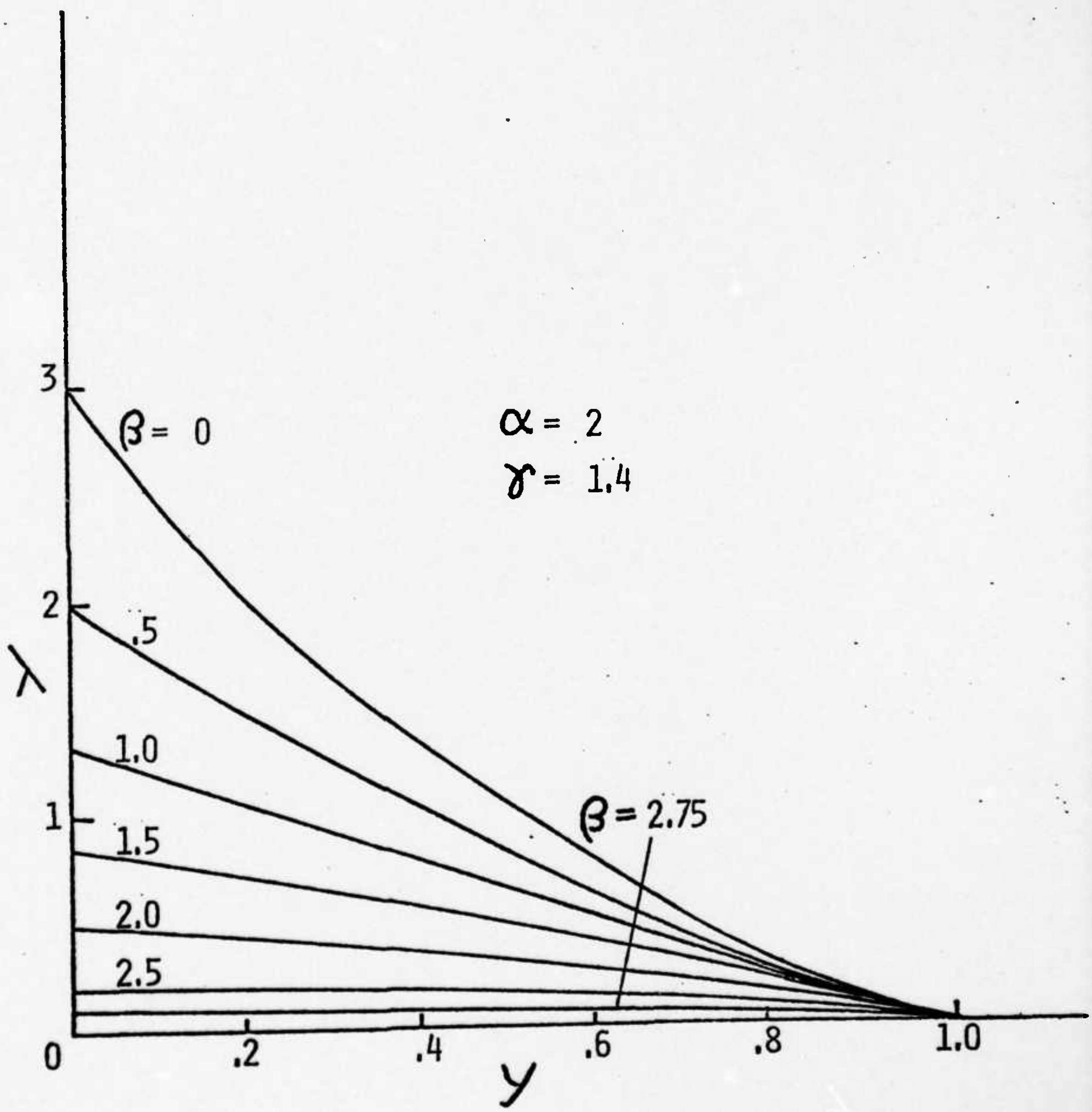
Flow Field Structure Behind Cylindrical Blast Wave for Constant Energy Input Rate.

Figure 6



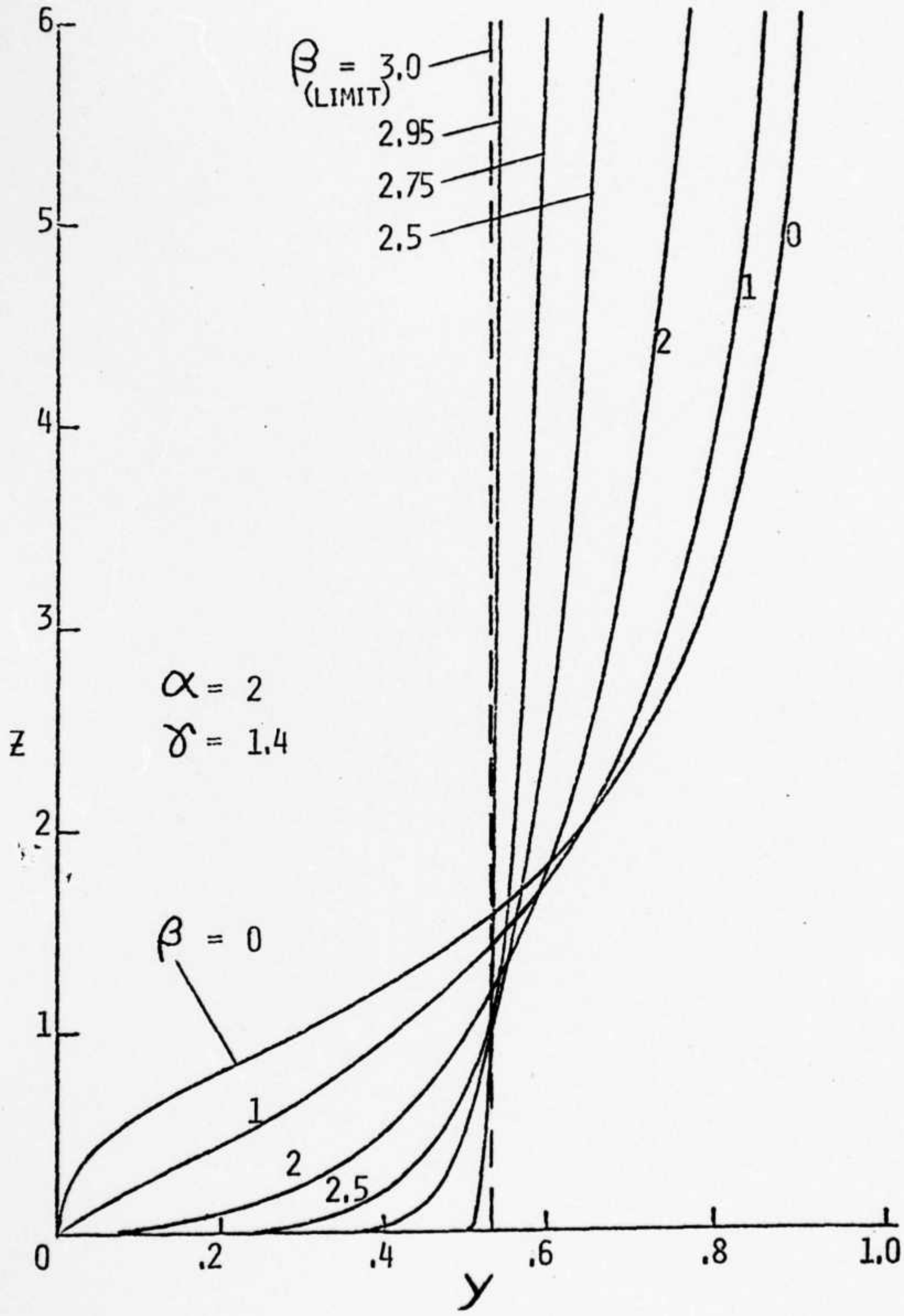
Decay Index and Shock Front Position from Second Order Perturbation Solution (P)
Compared with "Exact Solution" of Goldstine and Von Neumann (G)

Figure 7



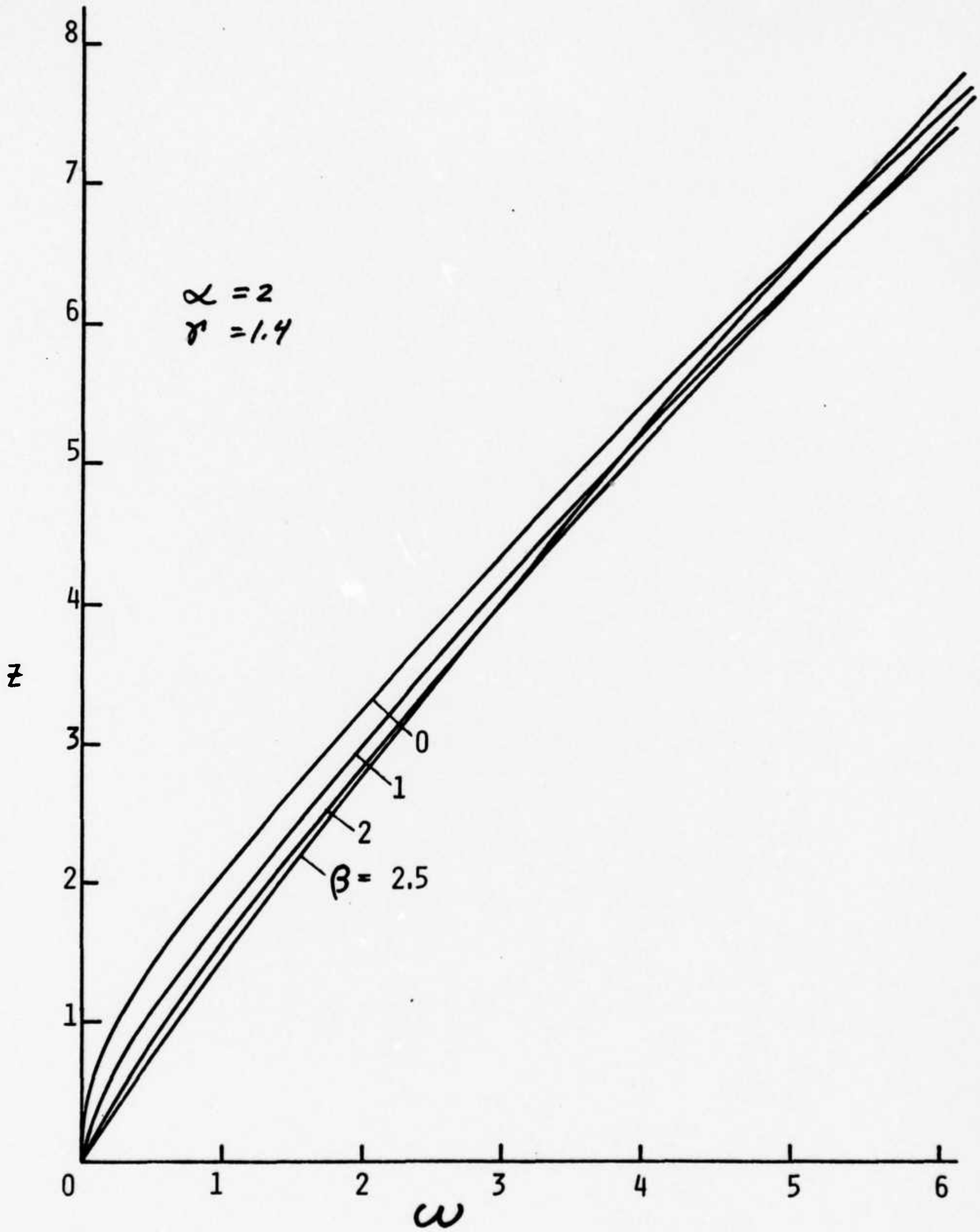
Variation of Decay Index with Shock Strength.

Figure 8



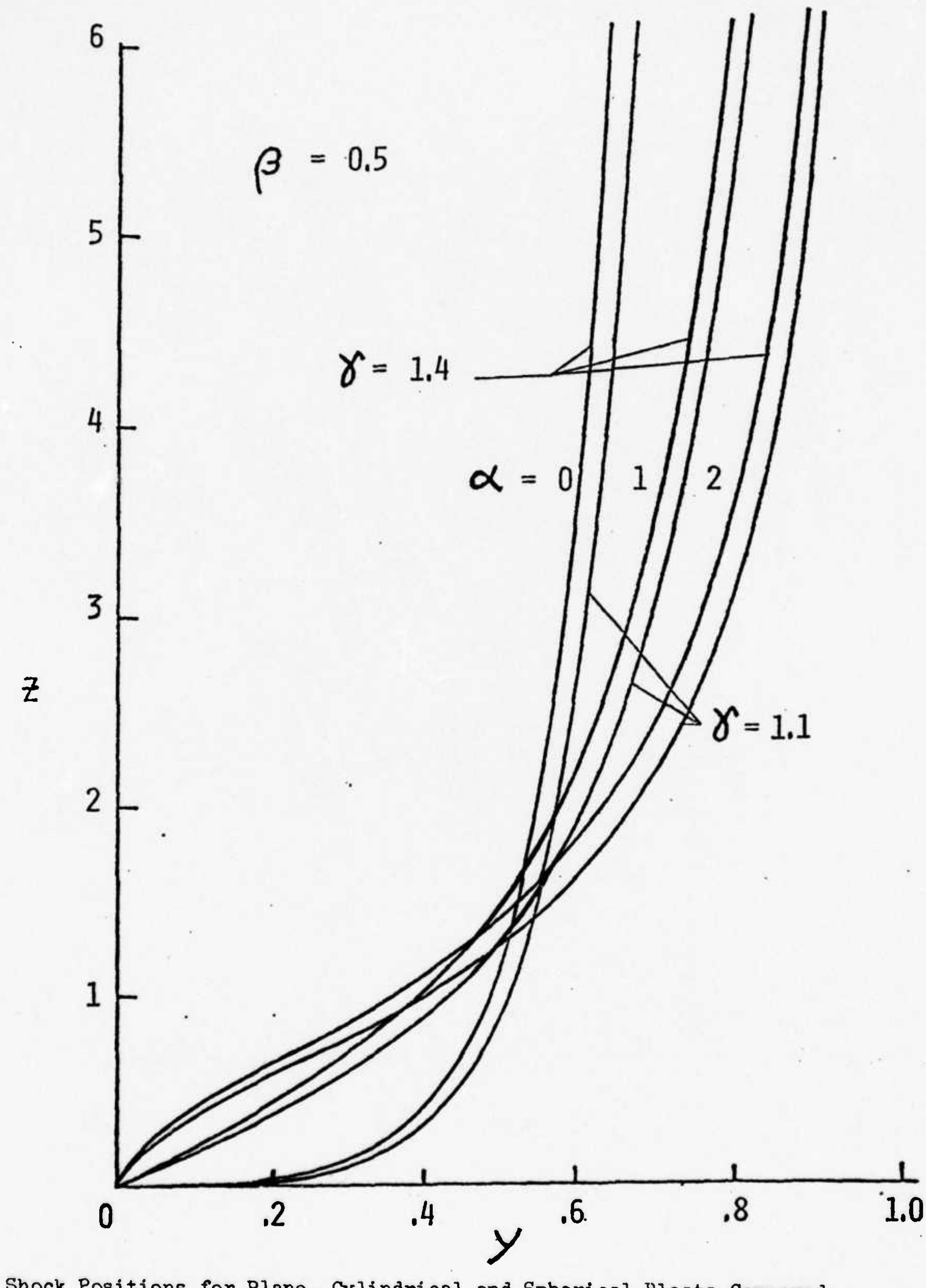
Shock Position as a Function of Shock Strength and Energy Input Parameter.

Figure 9



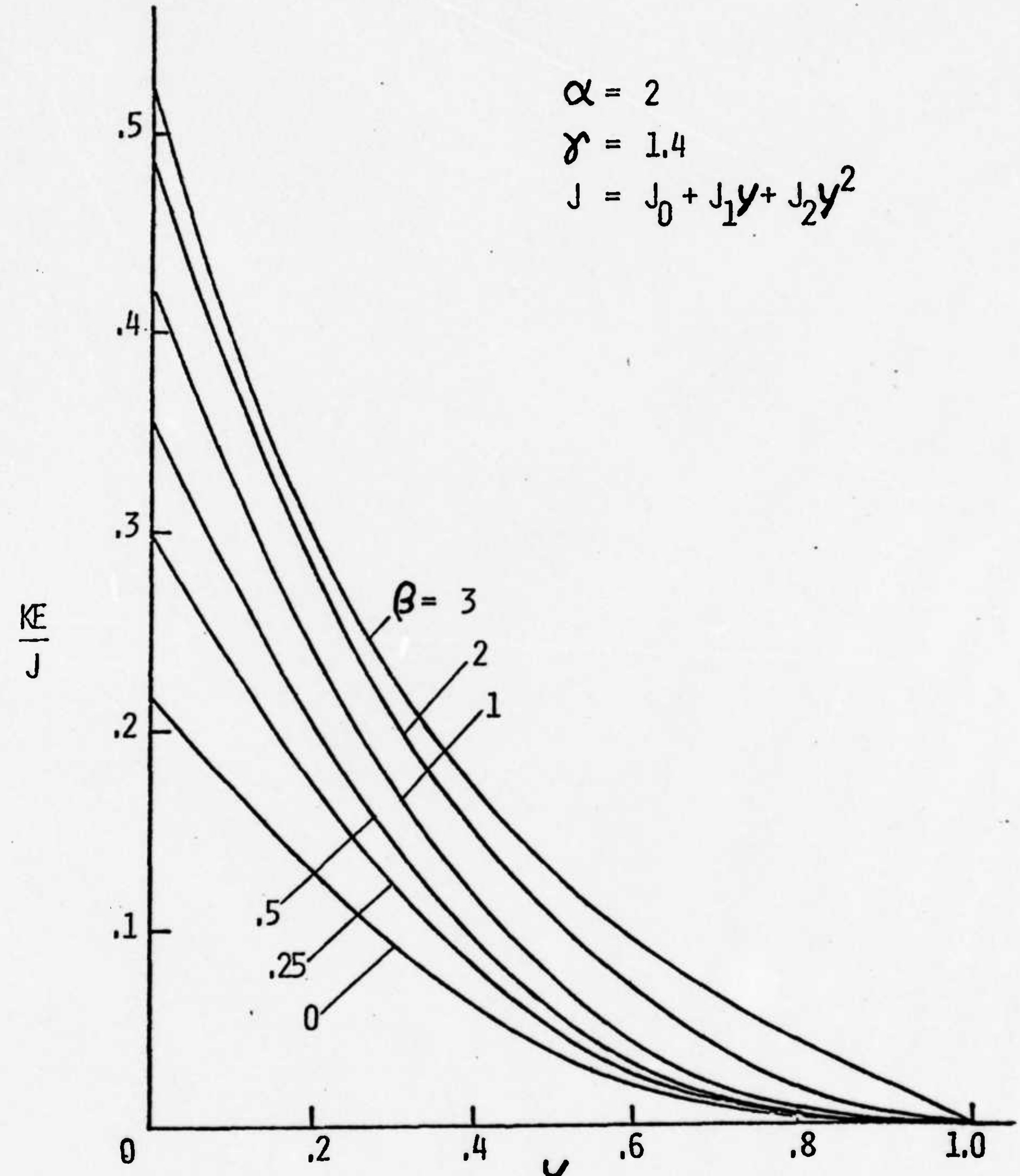
Shock Position as a Function of Time.

Figure 10



Shock Positions for Plane, Cylindrical and Spherical Blasts Compared.

Figure 11



Kinetic Energy Fraction Variation with Shock Strength and Energy Input Parameter.

Figure 12.

APPENDIX A

Summary of first and second order equations required for blast wave computation

FIRST ORDER:

(A-1) $K_o = \frac{2\gamma}{\gamma+1}\left[\frac{\gamma-1}{(\alpha+1)(\gamma+1)}\right]^{\gamma}$

(A-2) $K_{11} = 2\gamma/(\gamma-1) - (\gamma-1)/2\gamma$

(A-3) $e_2 = 1+(\gamma-1)\lambda_o/\gamma(\alpha+1)$

(A-4) $e_1 = 1 - \lambda_o/\gamma(\alpha+1)$

(A-5) $e_3 = 1 + (2\gamma-1)\lambda_o/\gamma(\alpha+1)$

(A-6) $g_o = K_o/(a_o^{\alpha}a_o')^{\gamma}m^{\lambda_o/(\alpha+1)}$

(A-7) $A_o = \gamma(\alpha+1)[a_o^{\alpha}g_o/a_o'-(\alpha+1)m^2]$

(A-8) $B_o = \gamma(\alpha+1)(2\alpha+\lambda_o)m/2$

(A-9) $C_o = \gamma[\alpha(\alpha+1)g_o a_o^{\alpha}a_o'/a_o^2 + \lambda_o/2]$

(A-10) $F = \gamma(a_o-(\alpha+1)ma_o')$

(A-11) $B_1 = B_o + \gamma(\alpha+1)\left[a_o^{\alpha}\frac{g_o a_o''}{(a_o')^2} - \frac{g_o'}{a_o'} - \alpha\frac{g_o}{a_o} - 2m\lambda_o\right]$

(A-12) $C_1 = C_o + \gamma\lambda_o^2/2 - \alpha(\alpha+1)(\gamma-1)a_o^{\alpha}g_o'/a_o$

(A-13) $D_1 = K_{11}a_o^{\alpha}m^{\lambda_o/(\alpha+1)}[(\alpha+1)g_o' + g_o\lambda_o/m]$

(A-14) $E_1 = [(\alpha+1)a_o^{\alpha}g_o' - D_1/K_{11}]/\lambda_o - F/2$

(A-15) $a_{11}'' = (B_1a_{11}' + C_1a_{11} + D_1)/A_o$

$$(A\text{-}16) \quad a''_{12} = (B_1 a'_{12} + C_1 a_{12} + E_1)/A_o$$

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$$(A\text{-}17) \quad J'_{11} = \frac{F}{(\alpha+1)}[(1+\lambda_o)a_{11} - (\alpha+1)m a'_{11}]$$

$$+ g_o a_o^{\alpha} a'_o \left[K_{11}\frac{m^{\lambda_o/(\alpha+1)}}{(\gamma-1)} - \left(\frac{a'_{11}}{a'_o} + \alpha\frac{a_{11}}{a_o}\right)\right]$$

$$(A\text{-}18) \quad J'_{12} = \frac{F}{(\alpha+1)}[(1+\lambda_o)a_{12} - (\alpha+1)m a'_{12}]$$

$$+ g_o a_o^{\alpha} a'_o \left[\frac{1-m^{\lambda_o/(\alpha+1)}}{(\gamma-1)\lambda_o} - \left(\frac{a'_{12}}{a'_o} + \alpha\frac{a_{12}}{a_o}\right)\right]$$

$$(A\text{-}19) \quad \frac{g_{11}}{g_o} = K_{11} m^{\lambda_o/(\alpha+1)} - \gamma\left(\frac{a'_{11}}{a'_o} + \alpha\frac{a_{11}}{a_o}\right)$$

$$(A\text{-}20) \quad \frac{g_{12}}{g_o} = \frac{1-m^{\lambda_o/(\alpha+1)}}{\lambda_o} - \gamma\left(\frac{a'_{12}}{a'_o} + \alpha\frac{a_{12}}{a_o}\right)$$

$$(A\text{-}21) \quad \Delta J_{11} = \int_o^{\delta} J'_{11}\,dm = \frac{F\delta}{(\alpha+1)}[(1+\lambda_o)a_{11} - (\alpha+1)\delta a'_{11}]_{\delta}$$

$$+ \left[g_o\left(\frac{K_o}{g_o}\right)^{1/\gamma}\right]_{\delta}\left[K_{11}\frac{\delta^{e_2}}{e_2(\gamma-1)} - \left(\frac{a'_{11}}{a'_o} + \alpha\frac{a_{11}}{a_o}\right)_{\delta}\frac{\delta^{e_1}}{e_1}\right]$$

$$(A\text{-}22) \quad \Delta J_{12} = \int_o^{\delta} J'_{12}\,dm = \frac{F\delta}{(\alpha+1)}[(1+\lambda_o)a_{12} - (\alpha+1)\delta a'_{12}]_{\delta}$$

$$- \left[g_o\left(\frac{K_o}{g_o}\right)^{1/\gamma}\right]_{\delta}\left(\frac{\delta^{e_2}}{e_2\lambda_o(\gamma-1)} + \left[\frac{a'_{12}}{a'_o} + \alpha\frac{a_{12}}{a_o} - \frac{1}{\lambda_o(\gamma-1)}\right]_{\delta}\frac{\delta^{e_1}}{e_1}\right)$$

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SECOND ORDER:

(A-23). $K_1 = K_{11} - \lambda_1/\lambda_o$

(A-24) $K_{21} = K_{11}\,\lambda_1/\lambda_o - 2\gamma/(\gamma-1)^2 - (\gamma-1)^2/8\gamma^2 - \lambda_1^2/2\lambda_o^2$

(A-25) $a_1 = a_{11} + \lambda_1 a_{12}$

(A-26) $g_1/g_o = g_{11}/g_o + \lambda_1 g_{12}/g_o$

(A-27)
$$\frac{g_1'}{g_o} = \frac{g_1 g_o'}{g_o^2} + \frac{K_1\lambda_o m^{\lambda_o/(\alpha+1)}}{(\alpha+1)m}$$
$$- \gamma\left[\alpha\left(\frac{a_1'}{a_o} - \frac{a_1 a_o'}{a_o^2}\right) + \frac{a_1''}{a_o'} - \frac{a_1' a_o''}{(a_o')^2}\right]$$

(A-28) $G = a_1(1+\lambda_o) - (\alpha+1)m a_1'$

(A-29) $B_2 = B_1 - 2(\alpha+1)\gamma m\lambda_o$

(A-30) $C_2 = C_1 + (1+5\lambda_o/2)\gamma\lambda_o$

(A-31)
$$D_2 = (\alpha+1)a_o^{\alpha}\left\{K_{21}\lambda_o\left[\frac{g_o'}{\lambda_o} + \frac{2g_o}{(\alpha+1)m}\right] m^{2\lambda_o/(\alpha+1)}\right.$$
$$- K_1\lambda_1\left[\frac{g_o'}{\lambda_o} + \frac{g_o}{(\alpha+1)m}\right] m^{\lambda_o/(\alpha+1)} + g_1'\left[\frac{g_1}{g_o} + \alpha\frac{a_1}{a_o}\right]$$
$$+ \frac{g_o'}{2}\left(\gamma\left[\alpha\frac{a_1^2}{a_o^2} + \left(\frac{a_1'}{a_o'}\right)^2\right] - \frac{g_1^2}{g_o^2} - \frac{\lambda_1^2}{\lambda_o^2}\right)$$
$$\left. + \gamma g_o\left(\frac{\alpha a_1}{a_o^3}[a_o a_1' - a_1 a_o'] + \frac{1}{(a_o')^3}[a_o' a_1' a_1'' - (a_1')^2 a_o'']\right)\right\}$$
$$+ \frac{\gamma\lambda_1}{2}[a_1(1+4\lambda_o) - 3(\alpha+1)m a_1'] + \frac{1}{2}(\alpha-1)\alpha(\alpha+1)a_1^2 g_o'$$

(A-32) $$E_2 = a_o^{\alpha}\left(\frac{(\alpha+1)g_o'}{2\lambda_o}\left[1-m^{2\lambda_o/(\alpha+1)}\right] - \frac{g_o}{m}m^{2\lambda_o/(\alpha+1)}\right) - \frac{F}{2}$$

(A-33) $$a_{21}'' = (B_2 a_{21}' + C_2 a_{21} + D_2)/A_o$$

(A-34) $$a_{22}'' = (B_2 a_{22}' + C_2 a_{22} + E_2)/A_o$$

(A-35) $$J_{21}' = \frac{1}{\alpha+1}[F(\lambda_1 a_1 + (1+2\lambda_o)a_{21} - (\alpha+1)ma_{21}') + \frac{\gamma}{2}G^2]$$

$$+ \frac{g_o a_o^{\alpha} a_o'}{\gamma-1}\left[\frac{a_{21}'}{a_o'} + \frac{\alpha a_{21}}{a_o} + \frac{\alpha a_1}{a_o}\left(\frac{a_1'}{a_o'} + \frac{(\alpha-1)a_1}{2a_o}\right) + \frac{g_{21}}{g_o}\right.$$

$$\left. + \frac{g_1}{g_o}\left(\frac{a_1'}{a_o'} + \frac{\alpha a_1}{a_o}\right)\right]$$

(A-36) $$J_{22}' = \frac{F}{\alpha+1}[(1+2\lambda_o)a_{22} - (\alpha+1)ma_{22}']$$

$$+ \frac{g_o a_o^{\alpha} a_o'}{\gamma-1}\left[\frac{a_{22}'}{a_o'} + \frac{\alpha a_{22}}{a_o} + \frac{g_{22}}{g_o}\right]$$

(A-37) $$\frac{g_{21}}{g_o} = \frac{1}{2}\left(\frac{g_1^2}{g_o^2} - \frac{\lambda_1^2}{\lambda_o^2}\right) - \gamma\left[\alpha\left(\frac{a_{21}}{a_o} - \frac{a_1^2}{2a_o^2}\right) + \frac{a_{21}'}{a_o'} - \frac{1}{2}\left(\frac{a_1'}{a_o'}\right)^2\right]$$

$$+ K_{21}m^{2\lambda_o/(\alpha+1)} - \frac{K_1\lambda_1 m^{\lambda_o/(\alpha+1)}}{\lambda_o}$$

(A-38) $$\frac{g_{22}}{g_o} = \frac{1}{2\lambda_o}\left(1-m^{2\lambda_o/(\alpha+1)}\right) - \gamma\left(\frac{\alpha a_{22}}{a_o} + \frac{a_{22}'}{a_o'}\right)$$

(A-39) $$\Delta J_{21} = \int_0^{\delta} J'_{21}\,dm = \frac{\delta}{\alpha+1}\left[F(\lambda_1 a_1 + (1+2\lambda_o)a_{21} - (\alpha+1)\delta a'_{21} + \frac{\gamma}{2}G^2\right]_{\delta}$$

$$+\left[\frac{g_o}{\gamma-1}\left(\frac{K_o}{g_o}\right)^{1/\gamma}\right]_{\delta}\left\{\frac{\delta^{e_1}}{e_1}\left[\frac{g_1}{g_o}\left(\frac{a'_1}{a'_o}+\frac{\alpha a_1}{a_o}\right)+\frac{\alpha a_1}{a_o}\left(\frac{a'_1}{a'_o}+\frac{(\alpha-1)a_1}{2a_o}\right)\right.\right.$$

$$\left.+\frac{1}{2}\left[\frac{g_1^2}{g_o^2}-\frac{\lambda_1^2}{\lambda_o^2}+\gamma\left(\frac{\alpha a_1^2}{a_o^2}+\left(\frac{a'_1}{a'_o}\right)^2\right)\right]\right]-(\gamma-1)\left(\frac{\alpha a_{21}}{a_o}+\frac{a'_{21}}{a'_o}\right)\Big]_{\delta}$$

$$\left.+\frac{K_{21}\delta^{e_3}}{e_3}-\frac{K_1\lambda_1}{\lambda_o}\frac{\delta^{e_2}}{e_2}\right\}$$

(A-40) $$\Delta J_{22} = \int_0^{\delta} J'_{22}\,dm = \frac{\delta}{\alpha+1}\left[F(a_{22}(1+2\lambda_o) - (\alpha+1)\delta a'_{22}\right]_{\delta}$$

$$+\left[\frac{g_o}{\gamma-1}\left(\frac{K_o}{g_o}\right)^{1/\gamma}\right]_{\delta}\left\{\frac{1}{2\lambda_o}\left(\frac{\delta^{e_1}}{e_1}-\frac{\delta^{e_3}}{e_3}\right)-(\gamma-1)\left(\frac{\alpha a_{22}}{a_o}+\frac{a'_{22}}{a'_o}\right)_{\delta}\frac{\delta^{e_1}}{e_1}\right\}$$

APPENDIX B - COMPUTER PROGRAMS

```
C<<<<<<<<<<<<<<<<<<<< BLAST WAVE,VARIABLE ENERGY >>>>>>>>>>>>>>>>>>>>>>>>>
C         PROGRAM CODED BY PROFESSOR E.T. PITKIN
C         MECHANICAL ENGINEERING DEPARTMENT
C         UNIVERSITY OF CONNECTICUT
C         STORRS, CONN., 06268
C<<<<<<<<<<<<<<<<<<<<< SECOND ORDER COMPUTATION >>>>>>>>>>>>>>>>>>>>>>>>>>
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 MASS,LAMN,LAM1,LAM2,LAM3,LAM4,KN,K1,LAMP1,LAMDA,LAMN2
      DIMENSION ZA11(200),ZAP11(200),ZA12(200),ZAP12(200),ZG11(200),
     1ZG12(200),ZA21(200),ZA22(200),ZAP21(200),ZAP22(200),ZG21(200),
     2ZG22(200),C(16),ZA(200),ZAP(200),ZM(200),ZG(200),BZ(100),OM(100)
      EQUIVALENCE (C(1),LAMN),(C(2),KN),(C(3),K1),(C(4),ALP),(C(5),GM),
     1 (C(6),GMOG),(C(7),AK11),(C(8),AK21),(C(9),GN),(C(10),RLAM),
     2 (C(11),LAM1),(C(12),ALFA),(C(13),GAMMA)
C CONSTRUCT INITIAL VALUES FOR INTEGRATION
      SAVE = -.005D0
    4 FORMAT(3F10.4,I5)
   10 READ(5,4,END=999) ALFA,BETA,GAMMA,KFIELD
      L = ALFA + .001D0
      LP = L+1
      ALP = ALFA + 1.D0
      GPL = GAMMA + 1.D0
      GM = GAMMA - 1.D0
      GMOG = GM/GAMMA
      AK11 = 2.D0/GMOG - GMOG/2.D0
      LAMN = 2.D0*(ALP-BETA)/(2.D0+BETA)
      LAMN2 = LAMN**2
      LAMP1 = LAMN + 1.D0
      KN = 2.D0*GAMMA*(GM/GPL/ALP)**GAMMA/GPL
      K1 = 0.D0
      F1 = 1.D0 - LAMN/ALP/GAMMA
      F2 = 1.D0 + GMOG*LAMN/ALP
      F3 = 1.D0 + (2.D0*GAMMA - 1.D0)*LAMN/GAMMA/ALP
      KPASS = 1
      NV = 12
      WRITE(6,40)
      WRITE(6,41) ALFA,BETA,GAMMA
   20 STEP = SAVE
      KK = DABS(.01D0/STEP) + .1D0
      DO 21 I=1,15
   21 Z(I) = 0.D0
      MASS = 1.D0
      Z(1) = 1.D0
      Z(2) = GM/GPL/ALP
      Z(5) = 2.D0/ALP/GPL
      J = 191
      ZM(J) = 1.D0
      ZA(J) = 1.D0
      ZAP(J) = Z(2)
      ZA11(J) = 0.D0
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```
      ZA12(J) = 0.D0
      ZA21(J) = 0.D0
      ZA22(J) = 0.D0
      ZAP11(J) = Z(5)
      ZAP12(J) = 0.D0
      ZAP21(J) = 0.D0
      ZAP22(J) = 0.D0
      ZG(J) = KN/Z(2)**GAMMA
      ZG11(J) = ZG(J)*(AK11 - GAMMA*Z(5)/Z(2))
      ZG12(J) = 0.D0
      ZG21(J) = 0.D0
      ZG22(J) = 0.D0
   25 WRITE(6,42) ZA(J),ZAP(J),Z(3),ZG(J),ZM(J)
      IF(KPASS.EQ.2) GO TO 30
      WRITE(6,43) ZA11(J),ZAP11(J),ZA12(J),ZAP12(J),Z(8),Z(9)
      IF(KPASS.EQ.1) GO TO 50
   30 WRITE(6,44) ZA21(J),ZAP21(J),ZA22(J),ZAP22(J),Z(14),Z(15)
   40 FORMAT(////)
   41 FORMAT(' ALFA =',F5.3,3X,'BETA =',F5.3,3X,'GAMMA =',F6.4,//)
   42 FORMAT(/,3X,'AO =',E10.4,3X,'AOPRIME =',E10.4,3X,'JO =',E11.5,3X
     1,'GO =',E11.5,3X,'MASS =',F10.3)
   43 FORMAT(/,3X,'A11=',E10.4,3X,'A11PRIME=',E10.4,3X,'A12=',E10.4,3X
     1,'A12PRIME=',E10.4,3X,'J11=',E11.5,3X,'J12=',E11.5,/)
   44 FORMAT(/,3X,'A21=',E10.4,3X,'A21PRIME=',E10.4,3X,'A22=',E10.4,3X
     1,'A22PRIME=',E10.4,3X,'J21=',E11.5,3X,'J22=',E11.5,/)
   45 FORMAT(/,' DJO =',F10.5,2X,'EKO =',F10.5,2X,'EFRACO =',F10.5)
   46 FORMAT(/,' RLAM =',F10.5,2X,'K21 =',F10.5)
   47 FORMAT(/,' DJ11 =',F10.5,2X,'DJ12 =',F10.5,2X,'LAM1 =',F10.5,2X,
     1'J1 =',F10.5,2X,'EK1 =',F10.5,2X,'K1 =',F10.5)
   48 FORMAT(/,' DJ21 =',F10.5,2X,'DJ22 =',F10.5,2X,'LAM2 =',F10.5,2X,
     1'J2 =',F10.5,2X,'EK2 =',F10.5)
C COMMENCE INTEGRATION
   50 DO 100 I=1,KK
  100 CALL RUNKUT(Z,MASS,C,STEP,NV)
      J = J-1
      FM1 = MASS**(LAMN/ALP)
      IF(KPASS.EQ.2) GO TO 98
      ZM(J) = MASS
      ZA(J) = Z(1)
      ZAP(J) = Z(2)
      ZG(J) = KN/((Z(1)**L*Z(2))**GAMMA*FM1)
      ZA11(J) = Z(4)
      ZA12(J) = Z(6)
      ZAP11(J) = Z(5)
      ZAP12(J) = Z(7)
      ZG11(J) = ZG(J)*(AK11*FM1 - GAMMA*(ALFA*Z(4)/Z(1)+Z(5)/Z(2)))
      ZG12(J) = ZG(J)*((1.D0-FM1)/LAMN-GAMMA*(ALFA*Z(6)/Z(1)+Z(7)/Z(2)))
   98 IF(KPASS.EQ.1) GO TO 99
      ZA21(J) = Z(10)
      ZA22(J) = Z(12)
      ZAP21(J) = Z(11)
      ZAP22(J) = Z(13)
      ZG1 = ZG11(J) + LAM1*ZG12(J)
      ZA1 = Z(4) + LAM1*Z(6)
```

```
      ZAP1 = Z(5) + LAM1*Z(7)
      ZG21(J) =(((ZG1/ZG(J))**2-RLAM**2)/2.D0 - GAMMA*(ALFA*Z(10)/Z(1)
     1 + Z(11)/Z(2) - (ALFA*(ZA1/Z(1))**2 + (ZAP1/Z(2))**2)/2.D0)
     2 + AK21*FM1**2 - K1*RLAM*FM1)*ZG(J)
      ZG22(J) = ((1.D0-FM1**2)/2.D0/LAMN - GAMMA*(ALFA*Z(12)/Z(1) +
     1 Z(13)/Z(2)))*ZG(J)
C     CUT DOWN STEP SIZE AS MASS APPROACHES ZERO
   99 IF(J.EQ.101) STEP = STEP/10.D0
      IF(J.EQ.101) GO TO 101
      IF(J.EQ.11) GO TO 101
      IF(J.GT.2) GO TO 50
  101 Z3 = -Z(3)
      WRITE(6,42) ZA(J),ZAP(J),Z3,ZG(J),ZM(J)
      IF(KPASS.EQ.2) GO TO 104
      Z8 = -Z(8)
      Z9 = -Z(9)
      WRITE(6,43)  ZA11(J),ZAP11(J),ZA12(J),ZAP12(J),Z8,Z9
      IF(KPASS.EQ.1) GO TO 105
  104 Z14 = -Z(14)
      Z15 = -Z(15)
      WRITE(6,44) ZA21(J),ZAP21(J),ZA22(J),ZAP22(J),Z14,Z15
  105 IF(J.GT.2) GO TO 50
C     COMPUTE EXTRAPOLATED VALUES FOR MASS = 0 WITH GO = CONSTANT
      ENSTP = -.00001D0
      DO 202 JK=1,2
      DO 200 I=1,90
  200 CALL RUNKUT(Z,MASS,C,ENSTP,NV)
      FM1 = MASS**(LAMN/ALP)
      GN = KN/((Z(1)**L*Z(2))**GAMMA*FM1)
      Z3 = -Z(3)
      WRITE (6,42) Z(1),Z(2),Z3,GN,MASS
      IF(KPASS.EQ.2) GO TO 199
      Z8 = -Z(8)
      Z9 = -Z(9)
      WRITE(6,43) Z(4),Z(5),Z(6),Z(7),Z8,Z9
      IF(KPASS.EQ.1) GO TO 201
  199 Z14 = -Z(14)
      Z15 = -Z(15)
      WRITE(6,44) Z(10),Z(11),Z(12),Z(13),Z14,Z15
  201 IF(JK.EQ.2) GO TO 202
      Z2S = Z(2)
      Z4S = Z(4)
      Z5S = Z(5)
      Z6S = Z(6)
      Z7S = Z(7)
      GNS = GN
      ENSTP = ENSTP/10.D0
      IF(KPASS.EQ.1) GO TO 202
      Z10S = Z(10)
      Z11S = Z(11)
      Z12S = Z(12)
      Z13S = Z(13)
  202 CONTINUE
      IF(KPASS.EQ.2) GO TO 206
```

```
      AN = Z(1)                                                         BLST1590
      ANL = AN**L                                                       BLST1600
      ANP = Z(2)                                                        BLST1610
      FAJ = AN -ALP*ANP*MASS                                            BLST1620
      GNO = (GN/GNS)*GN                                                 BLST1630
      GN = .7D0*GN + .3D0*GNO                                           BLST1640
      GNKF = GN*(KN/GN)**(1.D0/GAMMA)/GM                                BLST1650
      FJ1 = GNKF*MASS**E1/E1                                            BLST1660
      FJ2 = GNKF*MASS**E2/E2                                            BLST1670
      FJ3 = GNKF*MASS**E3/E3                                            BLST1680
      ANO = AN**LP - ALP*(GM/GN)*FJ1                                    BLST1690
      IF(ANO.LE.1.D-13) ANO=1.D-13                                      BLST1700
      ANO = ANO**(1.D0/ALP)                                             BLST1710
      ANPO = (Z(2)/Z2S)*Z(2)                                            BLST1720
      DJO = GAMMA*MASS*FAJ**2/ALP/2.D0 + FJ1                            BLST1730
      AJO = DJO - Z(3)                                                  BLST1740
      EKO = DJO - FJ1 - Z(10)                                           BLST1750
      EFRACO = EKO/AJO                                                  BLST1760
      AMASS = 0.D0                                                      BLST1770
      WRITE(6,42) ANO,ANPO,AJO,GNO,AMASS                                BLST1780
      WRITE(6,45) DJO,EKO,EFRACO                                        BLST1790
      A11 = Z(4)                                                        BLST1800
      AP11 = Z(5)                                                       BLST1810
      A12 = Z(6)                                                        BLST1820
      AP12 = Z(7)                                                       BLST1830
      A110 = (A11/Z4S)*A11                                              BLST1840
      A120 = (A12/Z6S)*A12                                              BLST1850
      AP110 = (AP11/Z5S)*AP11                                           BLST1860
      AP120 = (AP12/Z7S)*AP12                                           BLST1870
      G11 = GN*(AK11*FM1 - GAMMA*(ALFA*Z(4)/Z(1)+Z(5)/Z(2)))            BLST1880
      G12 = GN*((1.D0-FM1)/LAMN-GAMMA*(ALFA*Z(6)/Z(1)+Z(7)/Z(2)))       BLST1890
      EK11 = GAMMA*MASS*FAJ*(LAMP1*A11/ALP - MASS*AP11)                 BLST1900
      EK12 = GAMMA*MASS*FAJ*(LAMP1*A12/ALP - MASS*AP12)                 BLST1910
      DJ11 = EK11 + AK11*FJ2 -GM*(ALFA*A11/AN+AP11/ANP)*FJ1             BLST1920
      DJ12 = EK12 + (FJ1-FJ2)/LAMN - GM*(ALFA*A12/AN+AP12/ANP)*FJ1      BLST1930
      EK11 = EK11 - Z(11)                                               BLST1940
      EK12 = EK12 - Z(12)                                               BLST1950
      AZ8 = DJ11 - Z(8)                                                 BLST1960
      AZ9 = DJ12 - Z(9)                                                 BLST1970
      WRITE(6,43) A110,AP110,A120,AP120,AZ8,AZ9                         BLST1980
      ZM(1) = MASS                                                      BLST1990
      ZA(1) = AN                                                        BLST2000
      ZAP(1) = ANP                                                      BLST2010
      ZG(1) = GN                                                        BLST2020
      ZA11(1) = A11                                                     BLST2030
      ZA12(1) = A12                                                     BLST2040
      ZAP11(1) = AP11                                                   BLST2050
      ZAP12(1) = AP12                                                   BLST2060
      ZG11(1) = G11                                                     BLST2070
      ZG12(1) = G12                                                     BLST2080
      IF(KPASS.EQ.1) GO TO 210                                          BLST2090
  206 WRITE(6,42) ANO,ANPO,AJO,GNO,AMASS                                BLST2100
      A1 = A11 + LAM1*A12                                               BLST2110
      AP1 = AP11 + LAM1*AP12                                            BLST2120
```

```
      A21 = Z(10)
      A22 = Z(12)
      AP21 = Z(11)
      AP22 = Z(13)
      A210 = (A21/Z10S)*A21
      A220 = (A22/Z12S)*A22
      AP210 = (AP21/Z11S)*AP21
      AP220 = (AP22/Z13S)*AP22
      G1 = G11 + LAM1*G12
      EK21 = GAMMA*MASS*(FAJ*(A21*(LAMP1+LAMN) - ALP*MASS*AP21 + LAM1
     1*A1) + (LAMP1*A1 - ALP*MASS*AP1)**2/2.D0)/ALP
      DJ21 = EK21 + AK21*FJ3 - K1*RLAM*FJ2 + FJ1*((G1/GN)*(ALFA*A1/AN +
     1 AP1/ANP) + ((G1/GN)**2 - RLAM**2 + GAMMA*(ALFA*(A1/AN)**2 + (AP1
     2/ANP)**2))/2.D0 - GM*(ALFA*A21/AN+AP21/ANP) + ALFA*(A1/AN)*((ALFA
     3 -1.D0)*A1/AN/2.D0+AP1/ANP))
      EK22 = GAMMA*MASS*FAJ*(A22* (LAMP1+LAMN) - ALP*MASS*AP22)/ALP
      DJ22 = EK22 + (FJ1-FJ3)/LAMN/2.D0 - GM*(ALFA*A22/AN+AP22/ANP)*FJ1
      EK21 = EK21 - Z(8)
      EK22 = EK22 - Z(9)
      AZ14 = DJ21 - Z(14)
      AZ15 = DJ22 - Z(15)
      WRITE(6,44) A210,AP210,A220,AP220,AZ14,AZ15
      ZA21(1) = A21
      ZA22(1) = A22
      ZAP21(1) = AP21
      ZAP22(1) = AP22
      ZG21(1) = (((G1/GN)**2 - RLAM**2)/2.D0 - GAMMA*(ALFA*Z(10)/Z(1)
     1 + Z(11)/Z(2) - (ALFA*( A1/Z(1))**2 + ( AP1/Z(2))**2)/2.D0)
     2 + AK21*FM1**2 - K1*RLAM*FM1)*GN
      ZG22(1) = ((1.D0-FM1**2)/2.D0/LAMN - GAMMA*(ALFA*Z(12)/Z(1) +
     1 Z(13)/Z(2)))*GN
      IF(KPASS.EQ.2) GO TO 220
C COMPUTE LAMEDAS AND ENERGY INTEGRAL CONTIBUTIONS
  210 OMEGO = 2.D0/(LAMN+2.D0)
      OMEG1 = (2.D0*LAMN+2.D0)/(3.D0*LAMN+2.D0)
      LAM1 = (1.D0/GM/ALP-AZ8)/(AZ9+AJ0*(BETA*OMEG1/OMEGO-ALP)/LAMN2)
      RLAM = LAM1/LAMN
      K1 = AK11 - RLAM
      AK21 = AK11*RLAM - 2.D0/GMOG/GM - GMOG**2/8.D0 - RLAM**2/2.D0
      AJ1 = AZ8 + LAM1*AZ9
      EK1 = EK11 + LAM1*EK12
      WRITE(6,47) DJ11,DJ12,LAM1,AJ1,EK1,K1
      WRITE(6,46) RLAM,AK21
      NV = 15
      KPASS = 2
      GO TO 20
  220 BZ1 = -LAM1/LAMN2
      OMEG2 = (4.D0*LAMN+2.D0)/(5.D0*LAMN+2.D0)
      SA = (BETA*OMEG2/OMEGO - ALP)/2.D0
      SB = -AJ0*SA/LAMN2
      SA =AJ0*BZ1**2*(SA*LAMP1 + BETA*(BETA-1.D0)*(OMEG1/OMEGO)**2
     1/2.D0 + ALP*((ALFA+2.D0)/2.D0 - BETA*OMEG1/OMEGO))
      LAM2 = (SA - AZ14)/(AZ15 - SB)
      LAM4 =-(1.D0-BETA/ALP) *((ALFA+2.D0)/4.D0 - LAM2 - 2.D0*LAM1 -
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```
     1 3.D0*LAMN)
      LAM3 = -LAMN - LAM1 - LAM2 - LAM4
      AJ2 = AZ14 + LAM2*AZ15
      EK2 = EK21 + LAM2*EK22
      WRITE(6,48) DJ21,DJ22,LAM2,AJ2,EK2
C COMPUTE SHOCK TRAJECTORY
      WRITE(6,40)
      BZ0 = (((2.D0+BETA)/(ALP+3.D0))**BETA/AJ0)**(1.D0/(ALP-BETA))
      BZ(1) = BZ1
      BZ(2) = -(LAM2 + LAMP1*LAM1*BZ1)/2.D0/LAMN2
      BZ(3) = -(LAM3+(LAMP1+LAMN)*LAM1*BZ(2)+LAMP1*LAM2*BZ1)/3.D0/LAMN2
      BZ(4) = -(LAM4+(1.D0+3.D0*LAMN)*LAM1*BZ(3)+(LAMP1+LAMN)*LAM2*BZ(2)
     1+LAMP1*LAM3*BZ1)/4.D0/LAMN2
      DO 221 K=5,100
      AK = K
      BZ(K) = -((1.D0+(AK-4.D0)*LAMN)*BZ(K-4)*LAM4 + (1.D0+(AK-3.D0)
     1*LAMN)*BZ(K-3)*LAM3+(1.D0+(AK-2.D0)*LAMN)*BZ(K-2)*LAM2
     2 + (1.D0+(AK-1.D0)*LAMN)*BZ(K-1)*LAM1)/LAMN2/AK
  221 OM(K) = 1.D0/(1.D0+.5D0/(AK+1.D0/LAMN))
      OM(1) = OMEG1
      OM(2) = OMEG2
      OM(3) = 1.D0/(1.D0+.5D0/(3.D0+1.D0/LAMN))
      OM(4) = 1.D0/(1.D0+.5D0/(4.D0+1.D0/LAMN))
      Y = 0.D0
      DO 225 I=1,20
      SUMZ = 0.D0
      SUMO = 0.D0
      DO 222 K=1,100
      TERMZ = BZ(K)*Y**K
      TERMO = OM(K)*TERMZ
      IF(TERMO.LT.1.D-5.AND.TERMZ.LT.1.D-5) GO TO 223
      SUMZ = SUMZ + TERMZ
  222 SUMO = SUMO + TERMO
  223 FIRST = BZ0*Y**(1.D0/LAMN)
      BZS = (1.D0 + SUMZ)*FIRST
      OMEGA = (OMEGO + SUMO)*FIRST*DSQRT(Y)
      LAMDA = LAMN + LAM1*Y + LAM2*Y**2 + LAM3*Y**3 + LAM4*Y**4
      AJ = AJ0 + AJ1*Y + AJ2*Y**2
      EK = EK0 + EK1*Y + EK2*Y**2
      EFRAC = EK/AJ
      WRITE(6,440) Y,OMEGA,BZS,AJ,EFRAC,LAMDA
  225 Y = Y + .05D0
      WRITE(6,442) ALFA,BETA,AJ0,AJ1,AJ2,LAMN,LAM1,LAM2,LAM3,LAM4,
     1 AZ8,AZ9,AZ14,AZ15,GAMMA,SAVE
C COMPUTE FLOW FIELD
      IF(KFIELD.EQ.0) GO TO 10
      DO 350 II=1,9
      Y = 0.1D0*(II-1)
      WRITE(6,40)
      LAMDA = LAMN + LAM1*Y + LAM2*Y**2 + LAM3*Y**3 + LAM4*Y**4
      WRITE(6,445) Y,LAMDA
      WRITE(6,446)
      DO 350 KK=1,110
      IF(KK.EQ.1) GO TO 302
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      IF(MOD((KK-1),5).NE.0.AND.KK.LT.100) GO TO 350
      IF (KK.LT.92) GO TO 302
      IF(KK.GT.100) GO TO 301
      JJ = 101 - (KK-91)*10
      GO TO 303
  301 JJ = 111 - KK
      GO TO 303
  302 JJ = 192 - KK
  303 ZA1 = ZA11(JJ) + LAM1*ZA12(JJ)
      ZA2 = ZA21(JJ) + LAM2*ZA22(JJ)
      ZAP1 = ZAP11(JJ) + LAM1*ZAP12(JJ)
      ZAP2 = ZAP21(JJ) + LAM2*ZAP22(JJ)
      ZAT = ZA(JJ) + ZA1*Y + ZA2*Y**2
      ZAPT = ZAP(JJ) + ZAP1*Y + ZAP2*Y**2
      V = ALP*ZAT**L*ZAPT
      VEL = ZAT - ALP*ZM(JJ)*ZAPT + LAMDA*Y*(ZA1+2.D0*ZA2*Y)
      ZGT = ZG(JJ) + (ZG11(JJ) + LAM1*ZG12(JJ))*Y +(ZG21(JJ)+LAM2*
     1 ZG22(JJ))*Y**2
      IF(KK.GT.1) GO TO 305
      ZGTF = ZGT
      VF = V
      VELF = VEL
  305 RHO = VF/V
      ZGT = ZGT/ZGTF
      VELR = VEL/VELF
      WRITE(6,448) ZM(JJ),ZAT,ZGT,RHO,VELR
  350 CONTINUE
      GO TO 10
  440 FORMAT('  Y =',F6.3,'  OMEGA =',E11.5,'  Z =',E11.5,'  J =',F10.5
     1,'  KE/J =',F10.5,'  LAMDA =',F10.5)
  442 FORMAT(/,' SUMMARY',/,' ALFA=',F10.5,2X,'BETA=',F10.5,2X,'J0=',
     1F10.5,2X,'J1=',F10.5,2X,'J2=',F10.5,/,' LAM0=',F10.5,2X,'LAM1=',
     2F10.5,2X,'LAM2=',F10.5,2X,'LAM3=',F10.5,2X,'LAM4=',F10.5,/,' J11='
     3,F10.5,2X,'J12=',F10.5,2X,'J21=',F10.5,2X,'J22=',F10.5,/,' GAMMA='
     4,F10.5,2X,'STEP=',F10.5)
  445 FORMAT(/,' Y =',F5.2,2X,'LAMDA =',F10.5)
  446 FORMAT(4X,15X,'MASS',13X,'RADIUS',11X,'PRESSURE',12X,'DENSITY'
     1,11X,'VELOCITY',//)
  448 FORMAT(4X,5F19.5)
  999 STOP
      END
```

```
      SUBROUTINE FCT(Z,ZP,C,M)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 LAMN,KN,M,LAM1,K1,LAMP1
      DIMENSION Z(15),ZP(15),C(16)
C AND Z TRANSFERS
      IF(M.NE.1.DO) GO TO 10
      LAMN = C(1)
      KN = C(2)
      K1 = C(3)
      ALP = C(4)
      GAMM = C(5)
      GMCG = C(6)
      AK11 = C(7)
      AK21 = C(8)
      RLAM = C(10)
      LAM1 = C(11)
      ALF = C(12)
      GAM = C(13)
      LAMP1 = LAMN + 1.DO
      L = ALF + .01DO
   10 AN = Z(1)
      ANP = Z(2)
COMPUTE COMMON TERMS
      ANL = AN**L
      FM1 = M**(LAMN/ALP)
      GN = KN/(ANL*ANP)**GAM/FM1
      FAJ = AN - ALP*M*ANP
      AA = ALP*GAM*(ANL*GN/ANP - ALP*M**2)
COMPUTE ZERO ORDER DERIVATIVES
      B = ALP*GAM*(2.DO*ALF + LAMN)*M/2.DO
      CO = GAM*(LAMN/2.DO + GN*ALF*ALP*ANL*ANP/AN**2)
      ZP(1) = ANP
      ZP(2) = (B*ANP - CO*AN - GN*ANL*LAMN/M)/AA
      EKO = GAM*FAJ**2/ALP/2.DO
      ZP(3) = EKO + GN*ANL*ANP/GAMM
COMPUTE FIRST ORDER DERIVATIVES
      A11 = Z(4)
      AP11 = Z(5)
      A12 = Z(6)
      AP12 = Z(7)
      ANPP = ZP(2)
      GNP = -GN*(LAMN/ALP/M + GAM*(ALF*ANP/AN+ANPP/ANP))
      BB = ALP*GAM*(ANL*(GN*ANPP/ANP**2 - GNP/ANP - ALF*GN/AN)
     1 -2.DO*LAMN*M) + B
      CC = CO + GAM*LAMN**2/2.DO - GAMM*ALF*ALP*ANL*GNP/AN
      DD = AK11*FM1*ANL*(GNP*ALP + GN*LAMN/M)
      EE = (ANL*ALF*GNP - DD/AK11)/LAMN - FAJ*GAM/2.DO
      ZP(4) = AP11
      ZP(5) = (BB*AP11+CC*A11+DD)/AA
      ZP(6) = AP12
      ZP(7) = (BB*AP12+CC*A12+EE)/AA
      IF(K1.NE.0.DO) GO TO 100
      EK11 =  GAM*FAJ*(A11*LAMP1-ALP*M*AP11)/ALP
      EK12 =  GAM*FAJ*(A12*LAMP1-ALP*M*AP12)/ALP
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      ZP(8) = EK11 + GN*ANL*(AK11*ANP*FM1/GAMM-AP11-ALF*ANP*A11/AN)
      ZP(9) = EK12 + GN*ANL*(ANP*(1.D0-FM1)/GAMM/LAMN - AP12
     1 - ALF*ANP*A12/AN)
      ZP(10) = EK0
      ZP(11) = EK11
      ZP(12) = EK12
      RETURN
COMPUTE SECOND ORDER DERIVATIVES
  100 A1 = A11 + LAM1*A12
      AP1 = AP11 + LAM1*AP12
      APP1 = ZP(5) + LAM1*ZP(7)
      G1 = GN*(RLAM - GAM*(ALF*A1/AN + AP1/ANP) + K1*FM1 )
      GP1 = G1*GNP/GN + GN*(K1*LAMN*FM1/M/ALP - GAM*(ALF*(AP1/AN-A1
     1*ANP/AN**2) + (APP1/ANP-AP1*ANPP/ANP**2)))
      A21 = Z(10)
      AP21 = Z(11)
      A22 = Z(12)
      AP22 = Z(13)
      BBB = BB - 2.D0*ALP*GAM*M*LAMN
      CCC = CC + GAM*LAMN*(1.D0+2.5D0*LAMN)
      DDD = AK21*LAMN*FM1**2*(GNP/LAMN+2.D0*GN/ALP/M) - K1*LAM1*FM1*
     1(GNP/LAMN+GN/ALP/M) + GNP*(GAM*(ALF*(A1/AN)**2+(AP1/ANP)**2) -
     2(G1/GN)**2-RLAM**2)/2.D0+GP1*(G1/GN+ALF*A1/AN)+GAM*GN*(ALF*A1*
     3(AN*AP1-A1*ANP)/AN**3 + (ANP*AP1*APP1-AP1**2*ANPP)/ANP**3)
      DDD = DDD*ALP*ANL + GAM*LAM1*(A1*(1.D0+4.D0*LAMN) - 3.D0*ALP*M*
     1 AP1)/2.D0 + ALP*ALF*(ALF-1.D0)*A1**2*GNP/2.D0
      EEE = .5D0*ALP*(ANL*GNP*(1.D0-FM1**2)/LAMN + GAM*M*ANP) -
     1 ANL*GN*FM1**2/M - GAM*AN/2.D0
      ZP(10) = AP21
      ZP(11) = (BBB*AP21+CCC*A21+DDD)/AA
      ZP(12) = AP22
      ZP(13) = (BBB*AP22+CCC*A22+EEE)/AA
      G21 =(((G1/GN)**2-RLAM**2)/2.D0 + AK21*FM1**2 - RLAM*K1*FM1
     1 - GAM*(ALF*(A21/AN-(A1/AN)**2/2.D0) + AP21/ANP - (AP1/ANP)**2
     2/2.D0))*GN
      G22 = GN*((1.D0-FM1**2)/LAMN/2.D0 - GAM*(ALF*A22/AN+AP22/ANP))
      EK21 = GAM*(FAJ*(LAM1*A1 + (LAMP1+LAMN)*A21 - ALP*M*AP21)
     1 + (LAMP1*A1 - ALP*M*AP1)**2/2.D0)/ALP
      ZP(14) = EK21 +ANL*ANP*((ALF*A21/AN + AP21/ANP + (AP1/ANP+(ALF
     1-1.D0)*A1/AN/2.D0)*ALF*A1/AN)*GN+G1*(AP1/ANP+ALF*A1/AN)+G21)/GAMM
      EK22 = GAM*(FAJ*((LAMP1+LAMN)*A22 - ALP*M*AP22))/ALP
      ZP(15) = EK22 + ANL*ANP*(GN*(ALF*A22/AN+AP22/ANP) + G22)/GAMM
      ZP(8) = EK21
      ZP(9) = EK22
      RETURN
      END
```

```
      SUBROUTINE RUNKUT(Z,X,C,H,N)
CONSTANT STEP, FOURTH ORDER RUNGE-KUTTA INTEGRATION ALGORITHM
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Z(15),C(16),Y(15),RK1(15),RK2(15),RK3(15),RK4(15)
      H2 = H/2.D0
      CALL FCT(Z,RK1,C,X)
      DO 10 J=1,N
   10 Y(J) = Z(J) + H2*RK1(J)
      X = X+ H2
      CALL FCT(Y,RK2,C,X)
      DO 11 J=1,N
   11 Y(J) = Z(J) + H2*RK2(J)
      CALL FCT(Y,RK3,C,X)
      DO 12 J=1,N
   12 Y(J) = Z(J) + H *RK3(J)
      X = X+ H2
      CALL FCT(Y,RK4,C,X)
      DO 13 J=1,N
   13 Z(J) = Z(J) + H*(RK1(J)+2.D0*(RK2(J)+RK3(J))+RK4(J))/6.D0
      RETURN
      END
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